Cyclic Interference Alignment in Multi-User Communication Networks

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Contents

1 Introduction

In the year 1948, Claude Shannon established a mathematical theory in his seminal work [1] for the elementary description of information transmission over noisy communication systems. He provided a conceptual probabilistic model for a point-to-point channel between a single transmitter and a single receiver impaired by noise. A main part of his contribution was deriving and proving the maximal data-rate for reliable communication on a Gaussian point-to-point channel under fixed power and bandwidth constraints. But actually achieving the maximal data-rate in an implemented communication system has been exceedingly challenging over several decades. Since then, Shannon's work inspired many scientists and engineers to design elaborate channel codes that counteract transmission errors in order to achieve the maximal data-rate. Nowadays, it is indeed possible to communicate with data rates pretty close to the *Shannon-capacity* with reasonable effort [2].

However, in contemporary wireless communication systems there are multiple users sharing the same communication medium. In contrast to omnipresent noise, the resulting multi-user interference is not a naturally occurring phenomenon - it is rather a self-induced impairment caused by multiple simultaneous transmissions. The ongoing trend of increasing wireless communication traffic leads to a tight bottleneck of the maximally achievable data rates per user.

So far, most conventional signalling strategies (cf. [3], [4], [5]) basically avoid interference by *signal orthogonalization*, by *treating weak interference as noise* or by *cancelling strong interference*. In case of orthogonalization, the available resources (e.g., time, bandwidth, space) are elaborately partitioned among the users so that each user obtains a limited share of an ordinary interference-free point-to-point channel to convey their dedicated signals to the intended receivers, so that non-dedicated users are masked out. For instance, if the data rate is demanded to be equal among each user, it will scale reciprocally. In other words, each one of $K > 1$ users only achieves the $\frac{1}{K}$-th ratio of the sum-rate of a point-to-point channel. Whereas treating interference as noise is indeed optimal for the case of very weak interference power as shown in [6], [7] and [8], the approach clearly results in very low data rates for moderate and strong interference power, especially for a high number of users. Similarly, cancelling interference is most effective for very strong interference power only. The most relevant cases in between, i. e., for multi-user channels with moderate interference power or even generic channel conditions, the sum-capacity and the capacity region of the individual rates per user remains unsolved and highly challenging so far.

1.1 Capacity Approximation

Even for the most elementary 2-user interference channel, the exact capacity characterization is not fully known and proven yet. However, it turned out that it is worthwhile to approximate the channel capacity of multi-user channels with several different techniques (cf. [5]). Within the scope of this theses, especially the degrees-of-freedom (DoF) measure for capacity approximation at high signal-to-noise-ratio (SNR), is of particular interest.

Furthermore, to ease an accurate capacity approximation, simplified multi-user channel models are considered by focussing on the challenging effects of multiple signal interactions and by de-emphasizing the effects of noise. The *linear deterministic channel model (LDCM)* [9] provides a promising model that yet accurately approximates these effects in terms of the highly relevant Gaussian channel model. As a result, this approach supported and inspired several seminal works studying the capacity of various elementary channels in terms of the highly relevant Gaussian channel model. In some cases the capacity is even characterized within a limited number of bits (e. g., the 2-user interference channel [4] and the relay channel [9]). These derivations also utilize genie-based bounding techniques as developed in [10] and [4]. But when considering interference channels with more than two users, the optimal communication schemes and their capacity are far less understood. Note that approximating the capacity for channels approximated the LDCM becomes increasingly challenging for multiple users as well.

1.2 The Fundamental Principle of Interference Alignment

Multi-user interference is conventionally considered as a harmful and hence unwanted impairment that must be circumvented. A novel approach is to permit interfering signals by designing them cooperatively such that the aggregate interference signals are contained to a finite subspace at each user. This approach is called *interference alignment (IA)* and introduced in the seminal publications [3] and [11].

The key idea of IA is to optimally arrange the signalling-dimensions among different users such that the dimensions of interfering signals overlay, while the dedicated signals remain separate and hence decodable at the intended receivers. In the optimal case, the dimensions occupied by the interfering signals fully overlap, as if each user only experiences one virtual interferer in the system. Especially for a multi-user interference channel as depicted in Figure 1.1 for three users, the data rate per user approximately scales by $\frac{1}{2}$ w. r. t. the capacity of interference-free point-to-point links for each dedicated user pair. It is remarkable that this scaling is independent of the total number of users K. This is a substantial gain, when compared to an orthogonalized communication scheme. By applying IA, the aforementioned bottleneck can already be overcome for $K \geq 3$ communicating user pairs.

This theoretical idea of IA has opened the door to efficient communication schemes [12] with multiple users, multiple relays, bidirectional communications, etc. Nonetheless, there are still many challenging limitations in both theory and practice to cope

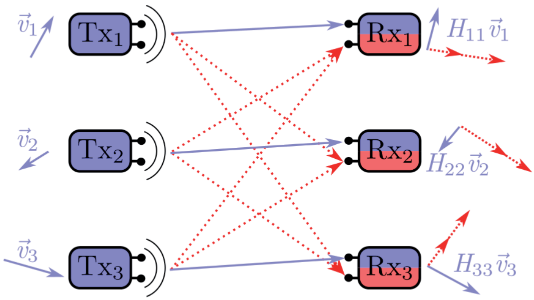


Figure 1.1: The fundamental principle of interference alignment: In a 3-user interference channel, 3 transmitters Tx_i communicate with 3 receivers Rx_i in pairwise distinct links. The interference of the 2 non-dedicated users is aligned to an interference subspace (red, dotted lines) at each receiver, while the dedicated signal (blue, solid lines) remains distinct.

with, e. g., the high demands on computational complexity and extremely accurate knowledge of the channel gains between all users.

A fundamental demand to enable IA - the *relativity of alignment* [12] - is that the channel itself must support a sufficient structure or variability so that an alignment scheme is feasible. Interestingly, IA is applied in a diverse number of substantially different schemes. It is enabled in multiple-input multiple-output (MIMO) channels by symbol-extensions in time or frequency [3], by rational independent dimensions [13], by signal-scale [14], on pairs of complementary matrices in ergodic channels [15], by propagation delay [3], [16], by (asymmetric complex) phase [17], on lattices [18], etc.

Clearly, IA has already drawn a lot of attention from the research areas of information theory, signal processing and even in distributed data-storage systems [19] and it still remains in the spotlight. For a more extensive overview and references on IA, we point the interested reader to the comprehensive survey [12].

1.3 Motivation

In this dissertation, our main focus is to introduce and develop a novel model for multi-user communication channels - the *cyclic polynomial channel model (CPCM)*. Inspired by the convenient algebraic framework of cyclic codes, our model enlightens a theoretical approach of IA using a polynomial ring. The CPCM is closely related to the widely accepted LDCM [9] and also motivated by the elementary example of *IA by propagation delay* given in [3]. A key difference to the LDCM is the use of a cyclic shift instead of a linear shift to model the impact of multiple interfering signals. From a mathematical perspective, such cyclic shifts uncover several valuable properties, that are hidden within the LDCM. It is a particular drawback of the LDCM that, for multi-user scenarios, the exponentially increasing number of parameters is hard to handle. In the literature, the authors mostly concentrate on multi-user channels with symmetrically parameterized channel gains, e. g., [20], [21], [22], or on arbitrary channel gains with only two user-pairs, e. g., [23], [24]. The proposed CPCM, however, is particularly tailored to tackle problems with asymmetric channel gains and multiple-users rather than directly focussing on the capacity-approximation within a limited number of bits.

1.4 Contributions

The CPCM supports the investigation of the following aspects:

In our initial research idea, the proposed CPCM was mainly dedicated to generalize the concept of *IA by propagation delay* for arbitrary channels, since unrolled cyclic shifts can be interpreted as propagation delays over the channel.

Moreover, the *capacity-achieving schemes* for the CPCM are inherently provided for *asymmetric channel gains* in multi-user systems. Deriving optimal schemes for the related LDCM is very challenging, since tracking the dependencies on arbitrary channel gains is immensely cumbersome if many users are involved.

While the conventional approach is to compute the achievable rates for a fixed channel, we turn the problem around: We fix a set of demanded rates and formulate *separability conditions* that must be satisfied by the channel to ensure decodability. These conditions are also useful explain particular singularities in the related LDCM.

Note that there is still a highly active and conflicting discussion about using *algebraic structure* versus *random codes* [25] in network and multi-user information theory. In this light, considering a model with cyclic shifts is quite amenable for an algebraic investigation.

In this thesis, we cover several different multi-user networks with the following properties:

Elementary unidirectional networks: The multiple-access channel, the broadcast channel, the interference channel and the X- channel. (Chapter 3)

Some users in the networks are equipped with backhaul links for constrained cooperation to provide a specific set of *cognitive messages*. (Chapter 3)

Relaying: Relays are auxiliary devices to enhance the channel by forwarding a function of the received signals and to enable some efficient schemes. (Chapter 4)

Multi-way communications: If users act as transceivers, they can simultaneously transmit and receive over a bidirectional channel. (Chapters 4 and 5)

In order to achieve the maximal DoF for these networks, our proposed communication schemes are mainly based on:

Orthogonal multiple-access: We show that the conventional othogonalization of signals is already optimal for some elementary channels. This is coherent with known results and intended to introduce the readers to the proposed model. (Chapter 3)

Linear coding and interference cancellation: However, these orthogonal schemes are already restrictive for interference channels. We derive linear coding strategies to resolve interference by cancellation. (Chapter 3)

Cyclic interference alignment: We develop cyclic IA schemes coherent to the fundamental IA principle described above. The provided capacity results are suitable for known results (e.g., for the X-channel and the K-user interference channel) of conventional channel models, in case they are already available. (Chapter 3)

Cyclic interference neutralization (IN): Especially in systems with multiple relays, there are multiple instances of the same interference from different relays. These signals can be aligned into the same subspace to neutralize each other over the air. (Chapter 4)

Network-coded cyclic signal alignment (SA): In bidirectional communications with relays, dedicated bidirectional signals are aligned so that a simple network code [26] is used at the receivers. (Chapters 4 and 5)

In the course of this dissertation, we do not only discuss DoF-achieving schemes for various channels, we also reveal several interesting phenomena and elaborate corresponding insights for multi-user systems:

(A) We observe different *duality relationships*: (Chapters 3, 4 and 5)

  (1) *Cyclic IA is dual to cyclic IN* in 3-user X-networks with a backhaul network of cognitive messages at the transmitters or receivers. (Chapter 3)

  (2) We observe a *user-relay duality*, for a cascaded chain of two-way relay networks, i. e., the roles of users and relays are swapped. It is linked to a *unicast-multicast*[1] *duality* in the uplink and its reciprocal downlink transmission. (Chapter 4)

  (3) We observe a *Δ-Y relationship* between 3-way channels and Y-channels, which is motivated by the well-known Δ-Y transformation in electrical circuit theory. (Chapter 5)

(B) We observe a *reciprocal alignment property* for a channel with two transmitters and two receivers: It basically states that aligning signals from the two transmitters at a primary receiver provides a particular interference pattern at the secondary receiver, while, vice versa, aligning two signals at the secondary receiver in an analogous way, provides the reciprocal interference pattern of the previous alignment at the primary transmitter. This property is exploited in the capacity-achieving scheme of the 2-user X-channel. (Chapter 3)

[1] In a *unicast* transmission, a single message is only dedicated for one receiver only. In a *multicast* transmission, a single messages is intended for multiple receivers at the same time.

(C) Moreover, we highlight several common properties that appear in Gaussian channels, MIMO channels and linear deterministic channels, likewise. Especially, constant MIMO channels exhibit a limitation caused by common eigenvectors when applying a perfect IA scheme. In the CPCM, an analogous property is also present for cyclic IA, but it only limits the feasibility of the communication system to a certain subset of channel matrices. (Chapters 3, 4 and 5)

Albeit, some of these phenomena, i. e., parts of (A)(1) and (C), were at least partially observed within the conventional channel models already, these insights extend the yet available knowledge and also serve as an additional confirmation for the validity of our proposed model.

Parts of this dissertation are published in [27], [28], [29], [30], [31], [32], [33] and [34].

1.5 Organization

In Chapter 2, we first recapitulate two basic models that are also treated in this thesis, before introducing the CPCM in detail. Then, optimal communication schemes for several elementary unidirectional communication channels are discussed in Chapter 3 using the CPCM. Furthermore, the 2-user X-channel, the 3-user X-channel, and the K-user interference channel are investigated for the application of cyclic IA. In Chapter 4, we extend the given model with relays and we apply cyclic IN. In a subsequent step,also two-way relaying and corresponding network-coded signal alignment schemes are included. In Chapter 5, we focus on 3-way communications that involve an elaborate combination of the methods discussed in the previous chapters. The 3-way channel is discussed in terms of the CPCM and in terms of the LDCM and Gaussian MIMO channel model (GMCM) for comparison. In Chapter 6, we briefly explore practical issues of IA in general and discuss IA by propagation delay in particular. The conclusion of this dissertation and directions for future works are provided in Chapter 7.

1.6 Notation

Symbol	Description
a, A	scalar/random variables (lower/upper case letters)
$\boldsymbol{a}, \boldsymbol{A}$	row vectors/matrices (boldface lower/upper case letters)
$\mathcal{A}$	sets (calligraphic upper case letters)
$\mathbf{0}_{a\times b}$	$a \times b$ zero matrix
$\boldsymbol{I}_{a\times b}$	$a \times b$ identity matrix
$\mathbf{1}_{a\times b}$	$a \times b$ one matrix
$\mathsf{span}(\boldsymbol{A})$	linear column-span of $\boldsymbol{A}$
$\mathsf{rank}(\boldsymbol{A})$	rank of $\boldsymbol{A}$
$\mathsf{dim}(\boldsymbol{A})$	dimension of $\boldsymbol{A}$
$\mathsf{det}(\boldsymbol{A})$	determinant of $\boldsymbol{A}$
$\boldsymbol{A}^{\mathsf{T}}$	transposed matrix of $\boldsymbol{A}$
$\boldsymbol{A}$	Moore-Penrose pseudo-inverse of a non-square $\boldsymbol{A}$
$\boldsymbol{A}^{-1}$	inverse matrix of $\boldsymbol{A}$
$\mathsf{diag}(a_1, ..., a_n)$	a square diagonal matrix with diagonal entries $a_1, ..., a_n$
$[a]^+$	$\max(0, a)$ positive part of $a \in \mathbb{R}$
$\binom{n}{k}$	binomial coefficient - n choose k
$H(X)$	entropy of a random variable X
$I(X;Y)$	mutual information between X and Y
$\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	complex normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$
x	an indeterminate, to address offsets in a polynomial
$p(x)$	a polynomial $\sum_{i=0}^{n-1} p^{[i]} x^i$ with coefficients $p^{[i]}$
Tx	transmitter (source)
Rx	receiver (destination)
T	transceiver (source/destination)
R	relay

Figure 1.2: Notation.

2 Basic Multi-User Communication Models

In this chapter, we will briefly recapitulate the preliminary properties of two widely established wireless channel models used in the literature on multi-user information theory:

The **Gaussian MIMO Channel Model (GMCM)**, and

the **Linear Deterministic Channel Model (LDCM)**.

Then, we introduce our novel channel model using polynomial rings:

The **Cyclic Polynomial Channel Model (CPCM)**[1].

We will elaborate several common properties between the CPCM and the two other channel models within the course of this thesis.

2.1 Common Parameters of the Considered Channel Models

The following common parameters and sets will be consistently used for all considered channel models unless it is stated otherwise. We consider multi-user interference channels with $K_{\mathrm{Tx}} \in \mathbb{N}$ transmitters and $K_{\mathrm{Rx}} \in \mathbb{N}$ receivers. Transmitters are denoted by Tx_i with index $i \in \mathcal{K}_{\mathrm{Tx}} = \{1, ..., K_{\mathrm{Tx}}\}$. Receivers are denoted by Rx_j for $j \in \mathcal{K}_{\mathrm{Rx}} = \{1, ..., K_{\mathrm{Rx}}\}$, respectively. A dedicated message from Tx_i to Rx_j is denoted by $\boldsymbol{w}_{ji}$ and has rate R_{ji} for $i \in \mathcal{K}_{\mathrm{Tx}}$, $j \in \mathcal{K}_{\mathrm{Rx}}$.

In case that transmitters and receivers are combined into a single device, it is called a transceiver T_i. We will further assume that transceivers are capable of perfect full-duplex operation. The set of transceiver-indices is denoted by $\mathcal{K}_{\mathrm{T}} = \{1, ..., K_{\mathrm{T}}\}$ for K_{T} transceivers T_i. A relay R_i, with index $i \in \mathcal{K}_{\mathrm{R}}$ is an auxiliary device that can both transmit and receive (and hence forward) messages. But in contrast to a transceiver, a relay does not initiate own messages. The set of relay-indices is denoted by $\mathcal{K}_{\mathrm{R}} = \{1, ..., K_{\mathrm{R}}\}$ for K_{R} relays.

2.2 Gaussian MIMO Channel Model (GMCM)

First, we briefly consider the *GMCM* describing multi-antenna systems with multiple users as considered in [35] for instance. We will discuss some results concerning *time-varying* MIMO channels in Section 4.1.4, and also results of the *time-constant* MIMO

[1]Parts of this work have been published in [27].

3-way channel in Sections 5.3 and 5.4. Note that the distinction between time-varying and time-constant MIMO channels is essential in the design of feasible IA schemes for multi-user MIMO interference channels.

2.2.1 GMCM: Definition

Let a transmitter Tx_i be equipped with an arbitrary number of $M_{\mathrm{Tx}_i} \in \mathbb{N}$ transmit antennas, and a receiver Rx_j with M_{Rx_i} receive antennas. The signal transmitted at time instance n from Tx_i is a vector $\boldsymbol{x}_i(n) \in \mathbb{C}^{M_{\mathrm{Tx}_i} \times 1}$ satisfying a power constraint P. The channel matrix for the MIMO channel from Tx_i to Rx_j is denoted by $\boldsymbol{H}_{ji} \in \mathbb{C}^{M_{\mathrm{Rx}_j} \times M_{\mathrm{Tx}_i}}$. These random channel matrices are generated i. i. d. from a continuous probability distribution. We assume full and global channel state information (CSI), i. e., all channel matrices are perfectly known at each user. Depending on the discussed communication problems in the later chapters, we will specify whether the channel coefficients are time-varying or constant throughout the whole communication. The received signal at Rx_j is a vector $\boldsymbol{y}_j(n) \in \mathbb{C}^{M_{\mathrm{Rx}_j} \times 1}$. $\boldsymbol{y}_j(n)$ is a superposition of the transmitted signals from each Tx_i, weighted by $\boldsymbol{H}_{ji}$, respectively, and i.i.d. complex additive white Gaussian noise (AWGN) $\boldsymbol{z}_j \sim \mathcal{CN}(\mathbf{0}_{M_{\mathrm{Rx}_j} \times 1}, \boldsymbol{I}_{M_{\mathrm{Rx}_j} \times M_{\mathrm{Rx}_j}})$:

$$\boldsymbol{y}_j(n) = \sum\nolimits_{i \in \mathcal{K}_{\mathrm{T}}} \boldsymbol{H}_{ji} \boldsymbol{x}_i(n) + \boldsymbol{z}_j(n). \tag{2.1}$$

2.2.2 Degrees-of-Freedom: A Capacity Approximation for the GMCM

It has been a very challenging task to characterize the maximal achievable rates, i. e., the exact sum-capacity and the capacity region, for most of such Gaussian MIMO multi-user channels. In order to yet obtain an sufficiently accurate insight into these central problems, a widely used approach is to describe the scaling behaviour of the channel capacity within the high SNR regime. Such an approximation of the sum-capacity $C_\Sigma(P)$ is represented by:

$$C_\Sigma(P) = \mathrm{DoF} \cdot \log(P) + o(P). \tag{2.2}$$

The DoF-measure[2] is defined in [3] and [37] as the *pre-log factor* of the achieved sum-capacity $C_\Sigma(P)$:

$$\mathrm{DoF} = \lim_{P \to \infty} \frac{C_\Sigma(P)}{\log(P)}. \tag{2.3}$$

The DoF become accurate at high SNR, i. e., for power $P \to \infty$ as the approximation error vanishes with $o(P) \to 0$.

We will particularly apply the GMCM in Subsection 4.1.4, and in Sections 5.3 and 5.4.

[2]The DoF are also known as multiplexing gain (cf. [35], [36]).

2.2.3 Relativity of Alignment: Feasibility Conditions

The *relativity of interference alignment* as discussed in [12] is an elementary property that must be satisfied for any IA scheme known up to now. This property basically demands that each receiver requires an independent and sufficiently different view of the channel to enable IA. For the IA schemes considered in [3], [37] and [38], this property is satisfied by either considering time-varying channel coefficients in the single antenna case, or by employing multiple antennas at each receiver, so that the provided schemes do not degenerate and invalidate the use of IA. Similarly, for the ergodic IA scheme in [15], this property is provided by time-varying channel coefficients generated by an ergodic process.

Enabling IA is also described by satisfying a set of *feasibility conditions* for multi-user MIMO channels with constant coefficients as discussed in [39] and [40]. These feasibility conditions demand that there is an interference free-space for a certain set of decodable dedicated signals while interference is completely eliminated after zero-forcing.

In order to ensure that each dedicated signal is linearly decodable at its desired receiver, we demand that the following properties for the proposed communication schemes hold:

Linear interference-free: All signals that are dedicated for the receiver are allocated to a subspace that is linear independent of the interfering signals which are dedicated for other receivers.

Linear decodability: Multiple dedicated signals for the same receiver must be received linearly independent within the dedicated signal subspace.

2.3 Linear Deterministic Channel Model (LDCM)

In order to approximate the capacity of Gaussian multi-user networks with more emphasis on the relative strength of the channel gains, a wireless channel can be characterized in the high-SNR regime by the *linear deterministic channel model* (LDCM). The model was introduced by Avestimehr, Diggavi and Tse[3] in [9] and [41] and in the dissertation of Avestimehr [42]. This model is a special class of the general deterministic channel model introduced by El-Gamal and Costa in [43].

One of the main goals of the LDCM is to simplify the analysis of the capacity of Gaussian relay and interference channels with multiple users. Its underlying concept is to eclipse the impact of noise and to highlight the influence of essential properties of wireless signal interaction. In particular the authors of [9] focus on the effects caused by *broadcasting*, *interference* and *signal-scale* in the shared medium.

Furthermore, the LDCM is a useful tool for deriving the generalized degrees-of-freedom (GDoF) as considered in [4], which is a more accurate measure of the approximate capacity than the DoF. For highly accurate capacity characterizations whose capacity approximation error is within a constant gap, a refined linear deterministic channel model using a lower triangular Toeplitz matrix is proposed in [44]. However,

[3]The LDCM is also called the ADT-model.

the GDoF-metric is not in the scope of this thesis and hence its discussion is omitted. The concept of the constant-gap capacity approximation is only briefly recapitulated.

2.3.1 LDCM: Definition

In the LDCM of [41, Section II], the transmitted signals of each user are described by a sequence of $q \in \mathbb{N}$ bits. Each bit is indexed from the least significant bit level 1 to the most significant bit level q in a $1 \times q$ input bit vector $\boldsymbol{x} = (x(q), ..., x(1))$. The q bit levels transmitted from Tx_i to Rx_j are linearly shifted down by $q - n_{ji}$ levels, so that a number of $n_{ji} \in \mathbb{N}$ bits remains above the noise threshold. This is a so-called *signal-scale* representation (*signal-scale*-property), since the q bit levels are scaled down at the receivers. These *bit-pipes* emulate the attenuation of the channel gains from a transmitter to a receiver.

The parameter q for the maximal level is determined by the strongest channel gain, i. e., $q = \max_{i \in \mathcal{K}_{\mathrm{Tx}}, j \in \mathcal{K}_{\mathrm{Rx}}}(n_{ji})$. For the linear shifting operation, a $q \times q$ shift matrix with ones on the lower side-diagonal and zeros otherwise is used:

$$\boldsymbol{S} = \begin{pmatrix} \mathbf{0}_{1 \times q} & 0 \\ \boldsymbol{I}_{q-1 \times q-1} & \mathbf{0}_{q-1 \times 1} \end{pmatrix}. \tag{2.4}$$

$\boldsymbol{S}$ is a *nil-potent* matrix, since its q-th power and higher powers yield $\boldsymbol{S}^q = \mathbf{0}_{q \times q}$. The linear shift matrix is not invertible and its determinant is $\mathsf{det}(\boldsymbol{S}) = 0$. By applying the shift matrix $q - n_{ji}$ times on $\boldsymbol{x}$ provides the following $1 \times q$ output bit vector $\boldsymbol{y}_j$:

$$\begin{aligned} \boldsymbol{y}_j^{\mathsf{T}} &= \boldsymbol{S}^{q-n_{ji}} \boldsymbol{x}_i^{\mathsf{T}} \\ &= (0, ..., 0, x_i(q), ..., x_i(q - n_{ji})). \end{aligned}$$

We obtain a linear down-shifted version of input $\boldsymbol{x}_i$ with $n_{ji} \leq q$ potentially non-zero bit levels remaining in $\boldsymbol{y}_i$. Bit levels that are shifted below the lowest index of $\boldsymbol{y}_i$ are truncated and lost. This truncation represents the influence of the noise threshold on the received signal. By extending this operation to multiple transmitters and receivers, the received signal yields a superposition of all transmitted signals, which are each shifted by $q - n_{ji}$:

$$\boldsymbol{y}_j^{\mathsf{T}} = \textstyle\sum_{i \in \mathcal{K}_{\mathrm{Tx}}} \boldsymbol{S}^{q-n_{ji}} \boldsymbol{x}_i^{\mathsf{T}}, \tag{2.5}$$

for $j \in \mathcal{K}_{\mathrm{Rx}}$, respectively. The signals of the transmitter are broadcast to each receiver (*broadcast*-property) and the receivers obtain superimposed bits of multiple transmitters (*interference*-property). An exemplary 2 -user multiple-access channel is depicted in Figure 2.1 in terms of the LDCM. In order to compute the capacity by means of the LDCM, upper bounds on the maximal number of conveyable bits must be derived. Furthermore, achievable coding schemes must be designed to verify these upper bounds.

2.3.2 Capacity Approximation

The strength of the channel gains in the LDCM and the complex Gaussian channel model are related by $n_{ji} = \lceil \log_2(P) \rceil^+$. For a point-to-point channel from Tx_1 to Rx_1,

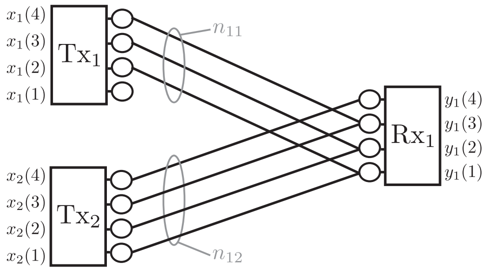


Figure 2.1: A linear deterministic multiple-access channel with two transmitters, one receiver, channel gains $n_{11} = 3$, $n_{12} = 4$, and a number of $q = 4$ bit levels is depicted. Most significant bits are the topmost and least significant bits are the lowermost bit levels.

the mutual information of the linear deterministic point to point channel is:

$$\begin{aligned} C_{\text{LDCM,P2P}} &= I(X_1; Y_1) = H(\boldsymbol{S}^{q-n_{11}}\boldsymbol{x}_1^{\mathsf{T}}) = H(\boldsymbol{y}_1^{\mathsf{T}}) && (2.6)\\ &= \textstyle\sum_{i=1}^{n_{11}} H(y_1(i) \mid y_1(i-1), ..., y_1(1)) && (2.7)\\ &= n_{11}. && (2.8) \end{aligned}$$

The entropy in the last line is maximized by the Bernoulli distribution with independent components with parameter $\frac{1}{2}$. This is a capacity approximation within a margin of at most one bit w. r. t. the complex Gaussian channel point-to-point capacity (cf. [1]) for high SNR:

$$\log_2(1+P) \leq n_{11} = \lceil \log_2(P) \rceil^+ \leq \log_2(2+2P) \leq \log_2(1+P)+1. \tag{2.9}$$

A well-known result as, e. g., given in [45, Section 15.3.6], is that the capacity region of the Gaussian multiple-access channel is characterized by the following upper bounds on the rates R_{11} and R_{12}:

$$\begin{aligned} R_{11} &\leq I(X_1; Y_1 \mid X_2) = \log_2(1+P_1), && (2.10)\\ R_{12} &\leq I(X_2; Y_1 \mid X_1) = \log_2(1+P_2), && (2.11)\\ R_{11}+R_{12} &\leq I(X_1, X_2; Y_1) = \log_2(1+P_1+P_2), && (2.12) \end{aligned}$$

for some input distribution $f_1(x_1)f_2(x_2)$ satisfying $\mathsf{E}(X_1^2) \leq P_1$ and $\mathsf{E}(X_1^2) \leq P_2$ for zero mean unit-variance real-valued Gaussian noise.

In case of the LDCM [41], the analogous representation of the capacity region, i. e., the closure of all achievable rate vectors, for the multiple-access channel is:

$$\begin{aligned} R_{11} &\leq I(X_1; Y_1 \mid X_2) = n_{11}, && (2.13)\\ R_{12} &\leq I(X_2; Y_1 \mid X_1) = n_{12}, && (2.14)\\ R_{11}+R_{12} &\leq I(X_1, X_2; Y_1) = \max(n_{11}, n_{12}), && (2.15) \end{aligned}$$

by computing the mutual information in terms of the LDCM, with $i \neq j \in \{1,2\}$:

$$\begin{aligned} I(X_i;Y_1 \mid X_j) &= H(Y_1 \mid X_j) - H(Y_1 \mid X_1, X_2) \\ &= H(Y_1 \mid X_j) \\ &= H(\boldsymbol{S}^{q-n_{11}}\boldsymbol{x}_1^{\mathsf{T}} + \boldsymbol{S}^{q-n_{12}}\boldsymbol{x}_2^{\mathsf{T}} \mid \boldsymbol{S}^{q-n_{1j}}\boldsymbol{x}_j^{\mathsf{T}}) \\ &= H(\boldsymbol{S}^{q-n_{1i}}\boldsymbol{x}_i^{\mathsf{T}}) \\ &= n_{1i}, \\ I(X_1, X_2; Y_1) &= H(Y_1) - H(Y_1 \mid X_1, X_2) = H(Y_1) \\ &= H(\boldsymbol{S}^{q-n_{11}}\boldsymbol{x}_1^{\mathsf{T}} + \boldsymbol{S}^{q-n_{12}}\boldsymbol{x}_2^{\mathsf{T}}). \\ &= \max(n_{11}, n_{12}). \end{aligned}$$

This capacity region can be achieved using orthogonal multiple-access schemes. Furthermore, the capacity region of the LDCM is actually within one bit of its Gaussian counterpart (cf. in [9] and [41]).

$$\begin{aligned} \max(\log_2(1+P_1), \log_2(1+P_2)) &\leq \log_2(1+P_1+P_2) \\ &\leq \max(n_{11}, n_{22}) \\ &= \max(\lceil \log_2(P_1) \rceil^+, \lceil \log_2(P_2) \rceil^+) \\ &\leq \max(\log_2(2+2P_1), \log_2(2+2P_2)) \\ &\leq \max(\log_2(1+P_1), \log_2(1+P_2)) + 1. \end{aligned}$$

The LDCM and its extensions have already gained a lot of interest by many researchers working in the realm of multi-user information theory. There is an enormous amount of works dedicated to the characterization the capacity of various multi-user channels based on the LDCM. Among these, we would like to point to other elementary examples in [9] and to some works on *signal-scale interference alignment* [20], [21], [22], [46], *interference neutralization* [47], [48], and *user-cooperation* [23].

Albeit the LDCM considerably simplifies the derivation of capacity-achieving schemes for symmetric channel gains and for small networks, it becomes increasingly challenging to compute the capacity of multi-user networks with non-symmetric channel gains. Furthermore, the capacity expressions derived for the LDCM exhibit some exceptional singularities, especially for fully-symmetric channels (cf. [21] and [22], for instance). In the following, we will introduce our main channel model of this dissertation that deals with the aforementioned challenging problems.

2.4 Cyclic Polynomial Channel Model (CPCM)

In contrast to the LDCM, which is essentially based on a linear shift of bit-vectors, the proposed CPCM is based on a polynomial ring with cyclic shifts within a finite number of dimensions. Our mathematical framework is inspired by algebraic cyclic codes as in [49] and by the related LDCM of [9] as discussed in the previous section. A particular benefit of the CPCM concerns the algebraically convenient description by polynomials. While conventional algebraic codes mainly focus on the detection and correction of errors caused by noise (cf. [50]) the CPCM is dedicated to characterize the effects of multi-user interference, even for arbitrary cyclic shifts.

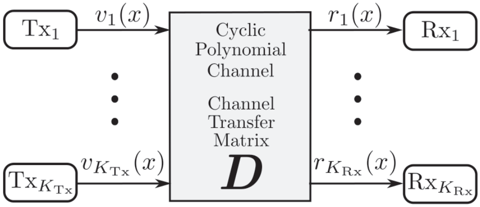


Figure 2.2: In a cyclic polynomial channel with K_{Tx} transmitters Tx_i and K_{Rx} receivers Rx_j, messages are conveyed by polynomials $v_i(x)$ over the channel by the transfer matrix $\boldsymbol{D}$. The polynomials $r_j(x)$ comprise the received messages.

We would like to remark, that a closely related channel model for multi-user line-of-sight (LoS) channels was introduced in [51]. Therein, similar effects of cyclic shifting are discussed as well. But in contrast to the approaches pursued in this thesis, the considered scheme in [51] is derived from a graph-theoretic perspective. Furthermore, another related model is investigated in [52] for an orthogonal frequency division multiplexing (OFDM) communication system in the time-frequency domain. The approach in [52] presents a channel decomposition with circulant matrices describing cyclic shifts.

2.4.1 CPCM: Definition

The CPCM provides a conceptual description of a wireless channel with $K_{\mathrm{Tx}} + K_{\mathrm{Rx}}$ users, i. e., K_{Tx} transmitters and K_{Rx} receivers, as depicted in Figure 2.2. We denote a source message to be conveyed from Tx_i to Rx_j, for $i \in \mathcal{K}_{\mathrm{Tx}}$ and $j \in \mathcal{K}_{\mathrm{Rx}}$, by $\boldsymbol{w}_{ji}$. Such a message $\boldsymbol{w}_{ji}$ comprises $m_{ji} \in \mathbb{N}_0$ subordinate messages (*submessages*) and is denoted by:

$$\boldsymbol{w}_{ji} = (W_{ji}^{[0]}, \ldots, W_{ji}^{[m_{ji}-1]}), \; W_{ji}^{[l]} \in \mathbb{F}, \tag{2.16}$$

for $l = 1, \ldots, m_{ji} - 1$. The total number of submessages m_{ji} may also be interpreted as rate demand between Tx_i and Rx_j. The communication channel has $n \in \mathbb{N}$ dimensions. The assignment of messages to these dimensions is represented by polynomials in the indeterminate x of degree $n - 1$. As for the well-known cyclic codes, we consider a commutative polynomial ring $\mathbb{F}(x)$ modulo $x^n - 1$ with coefficients over a field $\mathbb{F}$ (e.g., [53, Chapter 7 §2, p. 189]). The coefficients denote the carried information.

Encoding Scheme

All dedicated messages to be transmitted by Tx_i are encoded into the coefficients of the polynomial $v_i(x)$ by the encoding function e_i:

$$e_i: \; (\boldsymbol{w}_{1i}, \ldots, \boldsymbol{w}_{K_{\mathrm{Rx}}i}) \mapsto v_i(x).$$

The set of dedicated submessages is encoded in the coefficients of $v_i(x)$ with a linear code. A single position within the n dimensions[4] is addressed by an *offset* $x^0, x^1, \ldots, x^{n-1}$, from 0 (zero-offset) to $n - 1$ (maximal offset).

[4]The terms 'offsets' and 'dimensions' are synonymously used in this thesis.

Channel Transfer Matrix and Received Signal

An individual cyclic shift between Tx_i to Rx_j is denoted by the multiplication with a monomial $d_{ji} \in \mathcal{D}$ for $\mathcal{D} = \{x^k | k \in \mathbb{N}\} \cup \{0\}$. A zero-valued entry $d_{ji} = 0$ means that there is no link between Tx_i to Rx_j. The channel transfer matrix is assumed to be static and fully known to all users during the whole transmission, and it is defined by:

$$\boldsymbol{D} = (d_{ji})_{1 \leq j \leq K_{\mathrm{Rx}}, 1 \leq i \leq K_{\mathrm{Tx}}}. \tag{2.17}$$

The (cyclic polynomial) transfer function for K_{Tx} transmitters and K_{Rx} receivers is compactly described by a matrix multiplication of the vector of input polynomials $\boldsymbol{v}$ with the channel matrix $\boldsymbol{D}$, such that the vector of output polynomials $\boldsymbol{r}$ yields:

$$\boldsymbol{r}^{\mathsf{T}} \equiv \boldsymbol{D}\boldsymbol{v}^{\mathsf{T}} \bmod (x^n - 1). \tag{2.18}$$

with vectors $\boldsymbol{v} = (v_1(x), ..., v_{K_{\mathrm{Tx}}}(x))$ and $\boldsymbol{r} = (r_1(x), ..., r_{K_{\mathrm{Rx}}}(x))$. The modulo operation is taken element-wise. The received signal at Rx_j is a superposition of transmitted polynomials:

$$r_j(x) = \textstyle\sum_{i \in \mathcal{K}_{\mathrm{Tx}}} d_{ji} v_i(x) \bmod (x^n - 1). \tag{2.19}$$

Decoding Scheme

The received coefficients of the polynomials $r_j(x)$ are linearly decoded to obtain an estimate of the dedicated messages $\widehat{\boldsymbol{w}}_{ji}$. The decoding function for $r_j(x)$ is:

$$f_j : \quad r_j(x) \mapsto (\widehat{\boldsymbol{w}}_{j1}, ..., \widehat{\boldsymbol{w}}_{jK_{\mathrm{Rx}}}).$$

The dedicated messages are correctly decoded if $\hat{\boldsymbol{w}}_{ji} = \boldsymbol{w}_{ji}$ holds for all dedicated transmitter-receiver pairs. If a dedicated message $\boldsymbol{w}_{ji}$ cannot be linearly resolved, then there exists no error-free decoding scheme.

Example: Point-to-Point Channel

Consider the elementary point-to-point channel ($K_{\mathrm{Rx}} = K_{\mathrm{Tx}} = 1$) with $n \geq 1$ dimensions. Let only a single submessage W_{11} be transmitted at offset $x^{p_{11}}$ and cyclically shifted over the channel by $d_{11} = x^{\delta_{11}}$. For $v_1(x) = W_{11}x^{p_{11}}$ the polynomial received at Rx_1 is:

$$r_1(x) = d_{11} W_{11} x^{p_{11}} \equiv W_{11} x^{\delta_{11} + p_{11}} \mod (x^n - 1).$$

The dedicated message W_{11} is decoded at offset $x^{\delta_{11}+p_{11}} \bmod (x^n - 1)$.

2.4.2 Degrees-of-Freedom: Capacity of the CPCM

In order to evaluate the performance of a communication scheme over the given channel, we consider an analogous DoF-measure as defined in Section 2.2.2.

Assuming i. i. d. zero mean unit variance Gaussian noise at the receivers and an average power constraint P per message within each dimension, a single interference-free link between Tx_i and Rx_j has a capacity of $\log(P)$ bits per dimension at high

SNR. Recall from (2.3) that the DoF-measure is defined by:

$$\mathrm{DoF} = \lim_{P\to\infty} \frac{C_\Sigma(P)}{\log(P)}.$$

In the presented cyclic polynomial channel model, we define the achieved DoF by the total number of submessages M received interference-free per n dimensions:

$$\mathrm{DoF} = \lim_{P\to\infty} \frac{\frac{M}{n}\log(P)}{\log(P)} = \frac{M}{n}. \tag{2.20}$$

The messaging matrix[5] $\boldsymbol{M}$ denotes the number of demanded submessages dedicated to be conveyed from Tx_i to Rx_j per period of n dimensions:

$$\boldsymbol{M} = (m_{ji})_{1\le j\le K_{\mathrm{Rx}},1\le i\le K_{\mathrm{Tx}}},\ m_{ji} \in \mathbb{N}_0. \tag{2.21}$$

The total number of messages in the system is computed by:

$$M = \sum\nolimits_{j\in\mathcal{K}_{\mathrm{Rx}}} \sum\nolimits_{i\in\mathcal{K}_{\mathrm{Tx}}} m_{ji}. \tag{2.22}$$

From the number of dedicated and interfering submessages at each receiver, *upper bounds* on the DoF can be derived (e. g., Section 3.2). The **main task** is to design optimal encoding and decoding schemes for a particular setup of users and rate demands within the CPCM such that these upper bounds are achieved.

Recall the previous example: It is obvious that a messages $\boldsymbol{w}_{11}$ with $m_{11} = L$, $L \in \mathbb{N}$, submessages can be conveyed within at least $n \ge L$ dimensions in the point-to-point channel. Clearly, if these L submessages are each allocated to distinct dimensions, the single and merely shifted received signal is linearly decodable. If we choose $n = L$, then $\frac{M}{n} = \frac{L}{L} = 1$ DoF is always achievable. This result is in accordance with the DoF of the Gaussian point-to-point channel.

2.4.3 Relativity of Alignment: Separability Conditions

In terms of the CPCM, communication channels with multi-user interference are fully described by their channel matrices and their messaging matrices. Since we assume that the channel matrix is fully known to each user, one can always trace back all transmitter offsets that are incident to a particular receiver offset.

The proposed *separability conditions*[6] are slightly stricter than the feasibility conditions, which were briefly discussed in Section 2.2.3. In particular, we will further impose that the signals are interference-free and decodable *per each offset* and not only linear decodability. Hence, if one of the transmitter offsets contains a dedicated signal for a particular receiver offset, the incident transmitter offsets may not carry any desired or interfering signals and must remain silent. Otherwise, the dedicated signal would be subject to interference on that offset. In several cases these conditions are

[5]This matrix is basically a *unicast traffic matrix* as defined in [54], but without a normalization to one.

[6]The terminology is also used similarly in the discussion of real interference alignment in [13].

rather strict and effectively narrow the total number of channels down to the number of feasible channels.

The formulation of these separability conditions provides a useful tool to classify elementary properties of the channel matrix. We will elaborate the formal description of the separability conditions for the CPCM (cf. Sections 3.1, 4.1.1 and 4.2.3 in particular). As a result, we can determine the maximal number of achievable DoF and the feasibility of the proposed communication scheme based on the given channel matrix and the underlying message demands only. We will also discuss an interesting link between the separability conditions of cyclic IA on the CPCM and the feasibility conditions of MIMO IA on the GMCM in Section 3.4.3.

2.4.4 CPCM versus LDCM: An Algebraic Extension

There are various other ways to mathematically describe the cyclic shifts of the CPCM. Another option than using a polynomial ring to describe the behaviour of cyclic shifts, is to use vectors and powers of a particular circulant[7] $n \times n$ matrix:

$$\boldsymbol{T} = \begin{pmatrix} \mathbf{0}_{1\times n-1} & 1 \\ \boldsymbol{I}_{n-1\times n-1} & \mathbf{0}_{n-1\times 1} \end{pmatrix}. \tag{2.23}$$

Clearly, this matrix is almost identical to the linear shift matrix in (2.4) for the LDCM, except the top-right element. However, in contrast to the linear shift matrix $\boldsymbol{S}$, the circulant shift matrix $\boldsymbol{T}$ is not nil-potent, so that no integer exponent will produce a zero-matrix $\mathbf{0}_{n\times n}$. The cyclic shift circulant matrix is invertible and its determinant is $\mathsf{det}(\boldsymbol{T}) = 1$. $\boldsymbol{T}$ is also orthogonal, since $\boldsymbol{T}^\mathsf{T} = \boldsymbol{T}^{-1}$ holds.

Furthermore, we can describe the input signal from a transmitter Tx_i, $i \in \mathcal{K}_{\mathrm{Tx}}$, as an $1 \times n$ vector $\boldsymbol{v}_i = (v_i(0), ..., v_i(n-1))$. The output signal is an $1 \times n$ vector $\boldsymbol{r}_j = (r_j(1), ..., r_j(n-1))$ at Rx_j, which is a superposition of the input signals cyclically shifted by δ_{ji} positions:

$$\boldsymbol{r}_j^\mathsf{T} = \sum_{i=1}^{K_{\mathrm{Tx}}} \boldsymbol{T}^{\delta_{ji}} \boldsymbol{v}_i^\mathsf{T}, \tag{2.24}$$

for each $j \in \mathcal{K}_{\mathrm{Rx}}$. Since the offsets $x^0, ..., x^{n-1}$ may be interpreted as basis-vectors of an n dimensional vector space, we can also apply several useful operations from linear algebra like $\mathsf{rank}(\cdot), \mathsf{dim}(\cdot)$, etc., to this representation.

Each channel, based on the CPCM, has a specific counterpart in the LDCM (and vice versa). We replace the shift matrices $\boldsymbol{S} \leftrightarrow \boldsymbol{T}$ and hence the gains $\delta_{ji} \,(\mathrm{mod}\, n) \leftrightarrow n - n_{ji}$:

$$\boldsymbol{r}_j^\mathsf{T} = \sum_{i=1}^{K_{\mathrm{Tx}}} \boldsymbol{T}^{\delta_{ji}} \boldsymbol{v}_i^\mathsf{T}, \tag{2.25}$$

$$\boldsymbol{y}_j^\mathsf{T} = \sum_{i=1}^{K_{\mathrm{Tx}}} \boldsymbol{S}^{n-n_{ji}} \boldsymbol{x}_i^\mathsf{T}. \tag{2.26}$$

The LDCM basically provides a particular subset of bit-pipes of the LDCM. Thus, by using the cyclic shift operation, we highlight several algebraic properties that are concealed within the LDCM. Only if all channel parameters of the K-user interference

[7] A circulant $n \times n$ matrix $\boldsymbol{T}_n = (t_{j,i})_{0\le i,j\le n-1}$ has a certain symmetry described by $t_{j,i} = t_{i-j\,(\mathrm{mod}\,n)}$. Note that this is also a Frobenius companion matrix of the polynomial $\sum_{k=0}^{n-1} x^k$.

channel and the 2-user X-channel are fully-symmetric (i. e., $\delta_{ji} = 0$ for the CPCM, and $n_{ji} = q$ for the LDCM), then the channel matrices are raised by a power of zero. Each subchannel would degenerate to an $n \times n$ identity matrix $\boldsymbol{I}_{n\times n}$:

$$\boldsymbol{S}^0 = \boldsymbol{T}^0 = \boldsymbol{I}_{n\times n}. \tag{2.27}$$

As a result, both the CPCM and the LDCM exactly coincide in that case. This fully-symmetric parameterization is an interesting exception mentioned in the characterization of the linear deterministic and symmetric K-user interference channel in [22, Theorem 3.1 & Remark 1], and also for the linear deterministic 2-user X-channel in [21]. Singularities are observed in the sum-rate for fully-symmetric channel gains. For those cases, the authors of [22], [21] show that the achievable rates do not exceed the rates of an orthogonal multiple-access scheme.

Reconsidering this property from the viewpoint of the CPCM, these exceptional symmetries violate the separability conditions of the K-user interference channel (cf. Theorems 3.6 and 3.8) and also of the 2-user X-channel (cf. Theorem 3.9) as we will see in the next chapter. Hence, the CPCM does not only show an important analogy to the LDCM for this case, it also contributes to the analysis of the LDCM by providing a set of potential symmetry-candidates where the LDCM might display this exceptional behaviour. This insight also applies to more general channel symmetries. Furthermore, the proposed communication schemes derived from the CPCM might also indicate a first step showing how to align interference in the LDCM under general channel symmetries.

3 Multi-User Single-Hop Unidirectional Communications

In this chapter, we first consider the DoF of some elementary and well-known channels:

The K**-user multiple-access channel**,

the K**-user broadcast channel**, and

the 2 **-user interference channel**.

By applying the CPCM to these elementary channels, we introduce our basic methodology and validate known results on the DoF from a different perspective. With this approach, we also pursue a similar strategy as the authors of [35], [9], [13], [55], and [56], who also introduce new approximation measures or novel channel models for interference networks. In [35], the DoF-analysis of multi-user MIMO channels is introduced. In [9], the authors establish the LDCM. The idea of *real interference alignment* is presented in [13], and the *discrete superposition model* is introduced in [55]. A related finite-field channel model is introduced in [56]. In our next step, we discuss the DoF of three unidirectional channels based on the CPCM:

A general **upper bound**[1] on the DoF of a $K_{\mathrm{Tx}} \times K_{\mathrm{Rx}}$-user X-network,

the 2 **-user** X**- channel**[1],

the K**-user interference channel**[1], and

the 3 **-user** X**- network**[2].

Unidirectional Unicast Communications

In a *unidirectional unicast* communication system with multiple users, a message is dedicated to be conveyed from a single transmitter Tx_i to a single dedicated receiver Rx_i only. No message is dedicated for a group of multiple receivers as in the case of *multicast* communications. But nonetheless, a transmitter may transmit multiple different messages that are each dedicated for different receivers, and a receiver may also receive different dedicated messages from multiple transmitters.

[1]Parts of this work have been published in [27].
[2]Parts of this work have been published in [32].

3.1 Elementary Unidirectional Multi-User Channels

First, we apply the CPCM to well-known multi-user channels as the *multiple-access channel*, the *broadcast channel* and the 2-*user interference channel* and derive optimal communication schemes to achieve the maximal number of DoF within n dimensions, respectively.

3.1.1 Cyclic Polynomial Multiple-Access Channel

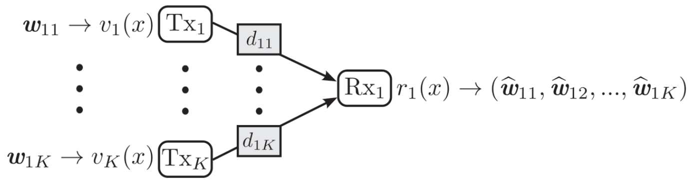


Figure 3.1: The cyclic polynomial multiple-access channel.

Our first example of a unicast communication system with multiple users is a multiple-access channel (MAC). In a K- user MAC, there are $K_{\mathrm{Tx}} = K$ transmitters and only $K_{\mathrm{Rx}} = 1$ receiver. An illustration of the MAC is depicted in Figure 3.1. In this case the messaging matrix degenerates to a $1 \times K$ row vector:

$$\boldsymbol{M} = (m_{11}, ..., m_{1K}). \tag{3.1}$$

Each Tx_i transmits a message $\boldsymbol{w}_{1i}$ that is dedicated for the single receiver Rx_1. The m_{1i} submessages from Tx_i to Rx_1 are denoted by $\boldsymbol{w}_{1i}$ and the corresponding offsets for the allocation in $v_i(x)$ are denoted by the parameters $p_i^{[l]}$, $l = 0, ..., m_{1i} - 1$, in the vector $\boldsymbol{p}_{1i}$:

$$\boldsymbol{w}_{1i} = (W_{1i}^{[0]}, ..., W_{1i}^{[m_{1i}-1]}), \tag{3.2}$$
$$\boldsymbol{p}_{1i} = (p_{1i}^{[0]}, ..., p_{1i}^{[m_{1i}-1]}). \tag{3.3}$$

The polynomials $v_i(x)$ allocate the m_{1i} submessages by:

$$v_i(x) = \textstyle\sum_{t=0}^{m_{1i}-1} W_{1i}^{[t]} x^{p_{1i}{}^{[t]}}. \tag{3.4}$$

Here, the vector of transmitted polynomials is $\boldsymbol{v} = (v_1(x), ..., v_K(x))$ and the vector of received polynomials degenerates to a single polynomial $\boldsymbol{r} = r_1(x)$. From (2.19), we obtain:

$$r_1(x) \equiv \textstyle\sum_{i \in \mathcal{K}} d_{1i} v_i(x) \mod (x^n - 1). \tag{3.5}$$

The submessages dedicated for Rx_1 must be linearly decodable. Hence, we demand that each submessage is effectively received in a separate dimension after the linear decoding procedure is performed. Thus, to keep all dedicated messages segregated

	x^0	x^1	x^2	x^3	x^4
$v_1(x)$	$W_{11}^{[0]}$	0	0	$W_{11}^{[1]}$	0
$v_2(x)$	$W_{12}^{[0]}$	0	0	0	0
$v_3(x)$	0	$W_{13}^{[1]}$	0	0	$W_{13}^{[0]}$
$r_1(x)$	$W_{13}^{[1]}$	$W_{11}^{[0]}$	$W_{12}^{[0]}$	$W_{13}^{[0]}$	$W_{11}^{[1]}$

Figure 3.2: This is a valid solution for orthogonal MA in a 3-user MAC for $n = 5$ dimensions, with $m_{11} = 2$, $m_{12} = 1$, and $m_{13} = 2$ submessages, and with cyclic 'right-shifts' $d_{11} = x$, $d_{12} = x^2$, and $d_{13} = x^4$. All separability conditions are fulfilled and Rx_1 can decode all five submessages achieving 1 DoF in total. The parameters are: $p_{11}^{[0]} = 0$, $p_{11}^{[1]} = 3$, $p_{11}^{[0]} = 0$, $p_{13}^{[0]} = 4$, $p_{13}^{[1]} = 1$.

at Rx_1, we demand that the following *multiple-access* conditions pairwise hold for all distinct $i \neq k \in \mathcal{K}_{\mathrm{Tx}}$:

$$d_{1i} x^{p_{1i}^{[l]}} \not\equiv d_{1k} x^{p_{1k}^{[t]}} \mod(x^n - 1), \tag{3.6}$$

for all $l \in \{0, ..., m_{1i} - 1\}$, $t \in \{0, ..., m_{1k} - 1\}$. Furthermore, the encoder must be linear. We demand that different submessages from the same Tx_i are also pairwise distinct:

$$d_{1i} x^{p_{1i}^{[l]}} \not\equiv d_{1k} x^{p_{1i}^{[t]}} \mod(x^n - 1), \tag{3.7}$$

for all $l, t \in \{0, ..., m_{1i} - 1\}$, $l \neq t$.

3.1.2 Orthogonal Multiple-Access (MA)

An orthogonal multiple-access (MA) scheme is a very basic allocation procedure for the offset-parameters $\boldsymbol{p}_{1i}$, for all $i \in \mathcal{K}_{\mathrm{Tx}}$. Each transmitted submessage is allocated at a distinct offsets at Rx_1. We will show that orthogonal MA achieves the following sum-DoF for $M = \sum_{i=1}^{K} m_{1i}$ submessages over $n = \sum_{k=1}^{K} m_{1k}$ dimensions:

$$\mathrm{DoF} \leq \frac{\sum_{i=1}^{K} m_{1i}}{\sum_{k=1}^{K} m_{1k}} = 1. \tag{3.8}$$

An exemplary encoding that uses an orthogonal MA scheme is depicted in the table of Figure 3.2. An orthogonal MAC scheme is already sufficient to achieve the upper bound on the DoF of the MAC.

Lemma 3.1. *The upper bound of* 1 *DoF in a MAC is achieved by orthogonal MA for* $n = \sum_{i=1}^{K} m_{1i}$ *dimensions.*

Proof:
(a) Necessity of $n \geq \sum_{i=1}^{K} m_{1i}$ *dimensions*:
Since all submessage must be linearly decodable at receiver Rx_1, the number of dimensions n is lower bounded by the total number of all submessages $n \geq \sum_{i=1}^{K} m_{1i}$. Otherwise dedicated messages would overlap at Rx_1 and can not be resolved linearly.

(b) Sufficiency of orthogonal MA to achieve 1 *DoF*:
We may assume any channel matrix $\boldsymbol{D}$ with arbitrary $d_{ij} \in \mathcal{D}$. W.l.o.g., we first fix all parameters of $\boldsymbol{p}_{11}$, such that all the $p_{11}^{[\cdot]}$ are pairwise distinct within the offsets $0, ..., n-1$. Now, a number of m_{11} offsets at Rx_1 is already used. Then, we iteratively fix the other offsets $\boldsymbol{p}_{1i}$ for $i = 2, ..., K$, so that they do not overlap with the already fixed offsets at receiver Rx_1. All submessages are orthogonal to each other within n dimensions. This is clearly feasible since all Tx_i can accommodate m_{1i} submessages at any offset within $n = \sum_{i=1}^{K} m_{1i}$ dimensions. Altogether, Rx_1 can decode all messages and the MA scheme achieves the upper bound of 1 DoF in total. ■

3.1.3 Cyclic Polynomial Broadcast Channel

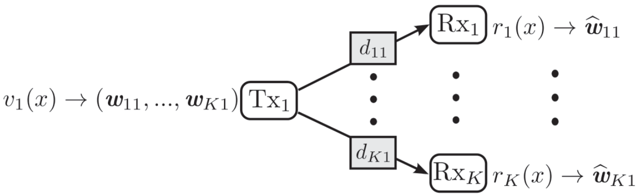


Figure 3.3: The cyclic polynomial broadcast channel.

An inverted example of the MAC is the cyclic polynomial broadcast channel (BC). The K-user BC has one transmitter Tx_1 and $K_{\mathrm{Tx}} = K$ receivers Rx_j, with $j \in \mathcal{K}_{\mathrm{Rx}}$. The BC is depicted in Figure 3.3. Transmitter Tx_1 intends to convey each message $\boldsymbol{w}_{j1}$ to its dedicated receiver Rx_j. The messaging matrix degenerates to a $K \times 1$ column vector:

$$\boldsymbol{M} = (m_{11}, m_{21}, ..., m_{K1})^{\mathsf{T}}. \tag{3.9}$$

Analogously to the MAC, we define the vectors for submessages and offset parameters:

$$\boldsymbol{w}_{j1} = (W_{j1}^{[0]}, ..., W_{j1}^{[m_{j1}-1]}), \tag{3.10}$$

$$\boldsymbol{p}_{j1} = (p_{j1}^{[0]}, ..., p_{j1}^{[m_{j1}-1]}). \tag{3.11}$$

The transmitted signal from Tx_1 is:

$$v_1(x) = \sum_{j \in \mathcal{K}_{\mathrm{Tx}}} \sum_{k=0}^{m_{j1}-1} W_{j1}^{[k]} x^{p_{j1}^{[k]}}. \tag{3.12}$$

The received signal at Rx_j is:

$$r_j(x) = d_{j1} v_1(x). \tag{3.13}$$

Messages, that are dedicated for receiver Rx_j but not for Rx_k, must be linear independent at Tx_1. Again we demand that the submessages are allocated to distinct dimensions. More precisely, the *intra-user interference conditions* for different receivers $j \neq k \in \mathcal{K}_{\mathrm{Rx}}$ are expressed by:

$$x^{p_{j1}^{[l]}} \not\equiv x^{p_{k1}^{[t]}} \mod (x^n - 1), \tag{3.14}$$

for $l \in \{0, ..., m_{j1} - 1\}$, $t \in \{0, ..., m_{k1} - 1\}$. The following lemma is basically analogous to Theorem 3.1.

Lemma 3.2. *The upper bound of* 1 *DoF in a BC is achieved by an orthogonal MA scheme for* $n = \sum_{i=1}^{K} m_{1i}$ *dimensions.*

Proof:
(a) Necessity of $n \geq \sum_{j=1}^{K} m_{j1}$ *dimensions*:
Since we demand a linear encoder, each submessage must already be linearly resolvable at the transmitter. Thus, the proof is basically analogous to Theorem 3.1 (a) for swapped indices.

(b) Sufficiency to achieve 1 *DoF by orthogonal MA*:
We assume a channel matrix $\boldsymbol{D}$ with arbitrary $d_{j1} \in \mathcal{D}$. W.l.o.g., we may fix $\boldsymbol{p}_{11}$ first, such that m_{11} offsets are used at Tx_1. Then, we iteratively fix the offsets $\boldsymbol{p}_{j1}$, for $j = 2, ..., K$, so that they do not overlap and remain orthogonal to the already fixed offsets at Tx_1.

The received signals $r_j(x)$ are merely cyclically shifted versions of the transmitted signal $v_1(x)$. In the received signals $r_j(x)$, the dimensions with non-dedicated messages are discarded and each dedicated submessage $\boldsymbol{w}_{j1}$ is decoded at receiver Rx_j. Thus, the intra-user interference conditions are satisfied by construction. Note that even all non-dedicated messages are decodable at each receiver. Altogether, the MA scheme achieves $\frac{M}{n} = \frac{\sum_{j=1}^{K} m_{j1}}{\sum_{j=1}^{K} m_{j1}} = 1$ DoF on the given BC. ■

3.1.4 Cyclic Polynomial 2-User Interference Channel

Yet another elementary unicast communication problem is described by the 2-user interference channel (2IFC). The corresponding Gaussian 2IFC is a fundamentally important object in the research area of multi-user information theory and has already been thoroughly discussed over several decades. However, note that the exact capacity characterization of the Gaussian 2IFC is still not known yet. Particularly important works on the 2IFC are [57], [4], [6], [7], [8], [58], [14], and [36], discussing interference cancellation, genie-aided upper-bounds, treating interference as noise, etc. The DoF are known to be upper bounded by 1 as shown in [36]. The capacity region of the linear deterministic 2IFC is provided in [59].

The cyclic polynomial 2IFC comprises $K_{\mathrm{Tx}} = 2$ transmitters and $K_{\mathrm{Rx}} = 2$ receivers, and thus we have $\mathcal{K}_{\mathrm{Rx}} = \mathcal{K}_{\mathrm{Tx}} = \mathcal{K}$. Tx_1 desires to convey a message $\boldsymbol{w}_{11}$ to Rx_1 with m_{11} submessages and concurrently, Tx_2 desires to convey $\boldsymbol{w}_{22}$ to Rx_2 with m_{22} submessages. This corresponds to a 2×2 diagonal messaging matrix:

$$\boldsymbol{M} = \mathsf{diag}(m_{11}, m_{22}). \tag{3.15}$$

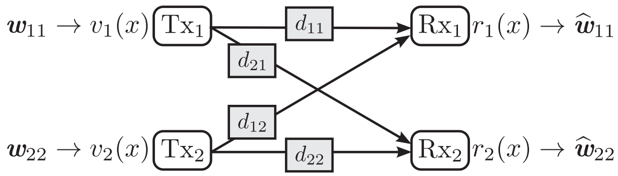


Figure 3.4: Messaging in the cyclic polynomial interference channel.

We have $M = m_{11} + m_{22}$ submessages in total. The channel matrix for the 2IFC is defined as:

$$\boldsymbol{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}. \tag{3.16}$$

The signals of each transmitter interfere at the undesired receivers over the cross-channels d_{12} and d_{21}. An illustration of the 2IFC is depicted in Figure 3.4. We denote the vector of submessages and of parameters for signals from Tx$_i$ to Rx$_i$, with $i \in \mathcal{K}$, by:

$$\boldsymbol{w}_{ii} = (W_{ii}^{[0]}, ..., W_{ii}^{[m_{ii}-1]}), \tag{3.17}$$
$$\boldsymbol{p}_{ii} = (p_{ii}^{[0]}, ..., p_{ii}^{[m_{ii}-1]}). \tag{3.18}$$

The transmitted signals from Tx$_1$ and Tx$_2$ are:

$$v_1(x) = \sum\nolimits_{k=0}^{m_{11}-1} W_{11}^{[k]} x^{p_{11}^{[k]}}, \tag{3.19}$$
$$v_2(x) = \sum\nolimits_{k=0}^{m_{22}-1} W_{22}^{[k]} x^{p_{22}^{[k]}}. \tag{3.20}$$

The polynomials received at Rx$_1$ and Rx$_2$ yield:

$$r_1(x) \equiv d_{11}v_1(x) + d_{12}v_2(x) \bmod (x^n - 1),$$
$$r_2(x) \equiv d_{21}v_1(x) + d_{22}v_2(x) \bmod (x^n - 1).$$

The 2IFC introduces another elementary type of interference: *inter-user interference*. We demand that the dedicated messages are received separately from the undesired interfering messages. In the 2IFC, there are only two *inter-user interference* conditions to consider:

$$d_{11}x^{p_{11}^{[l]}} \not\equiv d_{12}x^{p_{22}^{[t]}} \bmod (x^n - 1), \tag{3.21}$$
$$d_{21}x^{p_{11}^{[l]}} \not\equiv d_{22}x^{p_{22}^{[t]}} \bmod (x^n - 1), \tag{3.22}$$

for all $l \in \{0, 1, ..., m_{11} - 1\}$ and all $t \in \{0, 1, ..., m_{22} - 1\}$.

Lemma 3.3. *If* $\mathsf{det}(\boldsymbol{D}) \equiv 0 \bmod (x^n - 1)$ *holds, orthogonal MA is sufficient to achieve the upper bound of* 1 *DoF in the 2IFC.*

Proof:
(a) Necessity of $n \geq m_{11} + m_{22}$ *dimensions*:
At Rx$_1$, a number of m_{11} dedicated submessages must be linearly decodable and at

least m_{22} dimensions are occupied by interference. And vice versa at Rx_2, m_{22} dedicated submessages must be decodable and at least m_{11} dimensions are occupied by interference. Hence, at both receivers the necessary number of dimensions is lower bounded by $n = m_{11} + m_{22}$.

(b) Sufficiency of orthogonal MA to achieve 1 *DoF*:
The given condition on the determinant implies:

$$\begin{aligned} \mathsf{det}(\boldsymbol{D}) &\equiv 0 \bmod (x^n - 1) \\ \Leftrightarrow d_{11}d_{22} - d_{12}d_{21} &\equiv 0 \bmod (x^n - 1) \\ \Leftrightarrow d_{11}d_{12}^{-1} &\equiv d_{21}d_{22}^{-1} \bmod (x^n - 1). \end{aligned} \tag{3.23}$$

Due to this property, the separability conditions (3.21) and (3.22) coincide. We fix both parameters $\boldsymbol{p}_{11}$ and $\boldsymbol{p}_{22}$ such that orthogonality of their received offsets at Rx_1, and hence (3.21), is fulfilled. (3.22) will be fulfilled accordingly. As a result, all signals are received in orthogonal dimensions and both receivers can decode their dedicated signals (and even their interfering signals) within $n = m_{11} + m_{22}$ dimensions. ■

Clearly, if the converse statement $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod (x^n - 1)$ holds, a scheme purely based on orthogonal MA is not sufficient to achieve 1 DoF in the 2IFC exactly. A subset of signals will interfere and prohibit decodability of all signals. Thus $n > m_{11} + m_{22}$ dimensions are necessary if only orthogonal MA is applied.

There are two approaches to tackle this problem. The first approach is accomplished as follows. An orthogonal MA scheme that uses sufficiently long messages and a well-chosen guard interval is capable to asymptotically achieve 1 DoF. Let the cyclic shifts that are imposed by the channel be parameterized by δ_{ij}, i. e., we have $d_{ij} = x^{\delta_{ij}}$.

Corollary 3.4. *An orthogonal multiple-access scheme* ***asymptotically*** *achieves* 1 *DoF in the* 2 *-user interference channel for* $m_{11} + m_{22} \gg \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22}$.

Proof:
Let $n = m_{11} + m_{22} + 2\Delta_0$ with the guard interval $\Delta_0 = \delta_{12} + \delta_{22} + \delta_{11} + \delta_{21} \geq 0$. We first consider an allocation for Rx_1 and fix the parameters in $\boldsymbol{p}_{11}$ such that a continuous frame of m_{11} dimensions is allocated for $\boldsymbol{w}_{11}$ at Rx_1 within the dimensions $0, ..., m_{11}-1$. Then, we leave a guard interval of Δ_0 dimensions at Rx_1 and fix the parameters in $\boldsymbol{p}_{22}$ such that a continuous frame of m_{22} dimensions is allocated for $\boldsymbol{w}_{22}$ at Rx_1 within the dimensions $m_{11} + \Delta_0, ..., m_{11} + \Delta_0 + m_{22} - 1$. The remaining Δ_0 dimensions within $m_{11} + \Delta_0 + m_{22}, ..., m_{11} + \Delta_0 + m_{22} + \Delta_0 - 1$ are unused at Rx_1. Clearly, the signals at Rx_1 do not interfere within $n = m_{11} + \Delta_0 + m_{22} + \Delta_0$ dimensions for $\Delta_0 \geq 0$.

At Rx_2, this allocation also yields two frames of m_{11} and m_{22} dimensions, respectively. The frame of m_{11} dimensions occupies the dimensions:

$$0 - \delta_{11} + \delta_{21}, ..., m_{11} - \delta_{11} + \delta_{21}.$$

The frame of m_{22} dimensions occupies the dimensions

$$m_{11} + \Delta_0 - \delta_{12} + \delta_{22}, ..., m_{11} + \Delta_0 + m_{22} - \delta_{12} + \delta_{22}.$$

At Rx_2, the gap of unused frames on the 'right-hand side' of m_{11} occupies a number of:

$$\begin{aligned}\Delta_1 &= -\delta_{12} + \delta_{22} + m_{11} + \Delta_0 - (-\delta_{11} + \delta_{21} + m_{11}) \\ &= 2(\delta_{11} + \delta_{22}) \geq 0 \end{aligned} \tag{3.24}$$

dimensions. The second gap of unused frames on the 'right-hand side' of m_{22} has:

$$\begin{aligned}\Delta_2 &= -\delta_{11} + \delta_{21} + n - (-\delta_{12} + \delta_{22} + m_{11} + \Delta_0 + m_{22}) \\ &= 2(\delta_{12} + \delta_{21}) \geq 0 \end{aligned} \tag{3.25}$$

dimensions. As Δ_0, Δ_1 and Δ_2 are all non-negative, the signals do neither interfere at Rx_2. Altogether, for $m_{11} + m_{22} \gg 2\Delta_0 = \Delta_1 + \Delta_2$, the MA scheme asymptotically achieves the value one:

$$\mathrm{DoF} \leq \frac{M}{n} = \frac{m_{11} + m_{22}}{m_{11} + m_{22} + 2\Delta_0} \to 1. \tag{3.26}$$

■

We will see in the following, that 1 DoF is also exactly achievable for the case $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod (x^n - 1)$ by applying a *linear encoding and decoding (LEaD)* scheme with interference cancellation.

3.1.5 Linear Encoding and Decoding (LEaD)

In our previous considerations, the overlap of multiple signals is avoided beforehand by applying an orthogonal MA scheme so that each submessage is allocated to an individual dimension, respectively. This is independent of whether it is a dedicated or an interfering signal. Basically, an MA scheme treats undesired signals in the same manner as desired signals and then discards the interference afterwards. A LEaD scheme, however, provides the opportunity to decode overlapping interference from the received signal by using known side-information about interfering signals. In terms of the CPCM, the basic idea is to repeat the interfering signals within the yet unused and interference-free signal space of the undesired receiver. These repeated signals provide full side-information about the interference in order to cancel the overlapping interference from the dedicated signals. There are related LEaD schemes used for the LDCM (e. g., as in [59] and [21]). In the following, we define some auxiliary sets to describe the collection of dedicated offsets, of interfering offsets, etc.

The *set of all offsets* is defined by:

$$\mathcal{N} = \{0, ..., n-1\}. \tag{3.27}$$

The *set of dedicated offsets* from Tx_i allocated at Rx_j, for $i, j \in \mathcal{K}$, is defined by:

$$\mathcal{S}_{ji} = \{k \in \mathbb{N} \mid d_{ji} x^{p_{ji}^{[l]}} \equiv x^k \bmod (x^n - 1), l \in \{0, ..., m_{ji} - 1\}\}. \tag{3.28}$$

The *set of interfering offsets* from Tx_j received at Rx_j, for $i, j, t \in \mathcal{K}$, is defined by:

$$\mathcal{I}_{ji} = \{k \in \mathbb{N} \mid d_{ji} x^{p_{ti}^{[l]}} \equiv x^k \bmod (x^n - 1), l \in \{0, ..., m_{ti} - 1\}, t \neq j\}. \tag{3.29}$$

If an offset contains neither dedicated nor interfering signals, it is unused and empty. For the case of the 2IFC, the *set of unused offsets* at Rx_i is:

$$\mathcal{U}_i = \mathcal{N} \setminus (S_{ii} \cup \mathcal{I}_{ij}), \tag{3.30}$$

with $i \neq j \in \mathcal{K}$. Altogether, the particular sets of interest for the 2IFC are $\mathcal{N}$, $\mathcal{S}_{11}$, $\mathcal{S}_{22}$, $\mathcal{I}_{12}$, $\mathcal{I}_{21}$, $\mathcal{U}_1$ and $\mathcal{U}_2$.

Theorem 3.5. *A combined orthogonal multiple-access and LEaD scheme achieves the upper bound of* 1 *DoF of the 2IFC.*

Proof:
(a) Necessity of $n \geq m_{11} + m_{22}$ dimensions:
The proof of (a) is already provided in Lemma 3.3 (a). Moreover, note that:

$$n \leq m_{11} + m_{22} = |\mathcal{S}_{11}| + |\mathcal{I}_{12}| = |\mathcal{S}_{22}| + |\mathcal{I}_{21}| = |\mathcal{N}|. \tag{3.31}$$

(b) Sufficiency of orthogonal MA and LEaD to achieve 1 *DoF*:
For $\mathsf{det}(\boldsymbol{D}) \equiv 0 \mod (x^n - 1)$, achievability by orthogonal MA has already been proven in Lemma 3.3 (b). Hence, it suffices to consider the complementary case with $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod (x^n - 1)$.

W.l.o.g. we fix vectors $\boldsymbol{p}_{11}$ and $\boldsymbol{p}_{22}$ so that the inter-user interference condition in (3.21) holds, i. e., we apply an orthogonal offset allocation at Rx_1. Then, there is no overlapping interference at Rx_1 and we obtain the following properties:

$$\mathcal{S}_{11} \cap \mathcal{I}_{12} = \varnothing, \ \mathcal{S}_{11} \cup \mathcal{I}_{12} = \mathcal{N}, \ \mathcal{U}_1 = \varnothing. \tag{3.32}$$

Some offsets will violate (3.22) and overlap at Rx_2:

$$\mathcal{S}_{22} \cap \mathcal{I}_{21} \neq \varnothing, \ \mathcal{S}_{22} \cup \mathcal{I}_{21} \subset \mathcal{N}, \ \mathcal{U}_2 \neq \varnothing. \tag{3.33}$$

But, there are also unused offsets at Tx_2 in the non-empty set $\mathcal{U}_2$:

$$\mathcal{U}_2 = \mathcal{N} \setminus (\mathcal{S}_{22} \cup \mathcal{I}_{21}). \tag{3.34}$$

The number of unused offsets $|\mathcal{U}_2|$ and the number of overlapping offsets $|\mathcal{S}_{22} \cap \mathcal{I}_{21}|$ at Rx_2 is equal due to the inclusion-exclusion principle of united sets and due to (3.31):

$$\begin{aligned} |\mathcal{U}_2| &= |\mathcal{N}| - |\mathcal{S}_{22} \cup \mathcal{I}_{21}| = |\mathcal{N}| - (|\mathcal{S}_{22}| + |\mathcal{I}_{21}| - |\mathcal{S}_{22} \cap \mathcal{I}_{21}|) \\ &= |\mathcal{N}| - (|\mathcal{N}| - |\mathcal{S}_{22} \cap \mathcal{I}_{21}|) = |\mathcal{S}_{22} \cap \mathcal{I}_{21}|. \end{aligned} \tag{3.35}$$

In order to enable Rx_2 to cancel the overlapping interference from Tx_1 in $\mathcal{S}_{22} \cap \mathcal{I}_{21}$, Tx_1 repeats the overlapping part of its interfering signals within the offsets of $\mathcal{U}_2$. Now, Rx_2 cancels the known interference from Tx_1 and hence decodes its dedicated signal $\boldsymbol{w}_{22}$. Rx_1 cancels the repeated signals from Tx_1 using the decodable signals received in $\mathcal{S}_{11}$. This scheme corresponds to a simple linear decoding procedure. Both receivers can decode their dedicated and even the interfering signals. Altogether, the combined orthogonal MA and LEaD scheme achieves 1 DoF for a $\boldsymbol{D}$ with arbitrary $d_{ji} \in \mathcal{D}$. ∎

Up to this point, we have considered three elementary channels and introduced the corresponding separability conditions. We provided achievable schemes based on MA and LEaD. In the following, we focus on more extensive multi-user channels and we will also introduce cyclic IA in particular.

3.2 A General Upper Bound for $K_{\mathrm{Tx}} \times K_{\mathrm{Rx}}$ - User X-Networks

Before continuing with further unidirectional networks, we provide a more general upper bound on the DoF and combine the proof of the optimality for the theorems given in this chapter. Note that the general upper bound is basically analogous to [37, Theorem 1], albeit applied to the CPCM.

We denote the collection of dedicated messages received at Rx_j by the j-th row vector of $\boldsymbol{M}$:

$$\boldsymbol{\mu}_j = (m_{j1}, ..., m_{jK_{\mathrm{Tx}}}), \tag{3.36}$$

and we obtain $\boldsymbol{M} = (\boldsymbol{\mu}_1^\mathsf{T}, ..., \boldsymbol{\mu}_{K_{\mathrm{Rx}}}^\mathsf{T})^\mathsf{T}$. The total number of dedicated messages to be conveyed from all transmitters to Rx_j corresponds to summing up the entries in $\boldsymbol{\mu}_j$:

$$\|\boldsymbol{\mu}_j\|_1 = \textstyle\sum_{i=1}^{K_{\mathrm{Tx}}} m_{ji}.$$

The collection of messages transmitted from Tx_i is given by the i-th column vector of $\boldsymbol{M}$:

$$\boldsymbol{\nu}_i^\mathsf{T} = (m_{1i}, ..., m_{K_{\mathrm{Rx}}i}),$$

so that $\boldsymbol{M} = (\boldsymbol{\nu}_1, ..., \boldsymbol{\nu}_{K_{\mathrm{Tx}}})$. The total number of messages transmitted from Tx_i is the sum of elements in $\boldsymbol{v}_i$, respectively:

$$\|\boldsymbol{\nu}_i^\mathsf{T}\|_1 = \textstyle\sum_{j=1}^{K_{\mathrm{Rx}}} m_{ji}.$$

In general, the *multiple-access interference conditions* demand separability of all $\|\boldsymbol{\mu}_j\|_1$ desired messages, so that these messages are received interference-free at Rx_j. Furthermore, the *intra-user interference conditions* demand separability of all $\|\boldsymbol{\nu}_i^\mathsf{T}\|_1$ messages transmitted from Tx_i. For a pair $(\mathrm{Rx}_j, \mathrm{Tx}_i)$, there must be at least:

$$\|\boldsymbol{\mu}_j\|_1 + \|\boldsymbol{\nu}_i^\mathsf{T}\|_1 - m_{ji}$$

dimensions to ensure the two given separability conditions hold. Since entry m_{ji} appears twice when adding the 1-norm of the corresponding row vectors and column vectors, it must be subtracted once.

To further ensure that the *inter-user interference* conditions hold, the maximum of the sum $\|\boldsymbol{\mu}_j\|_1 + \|\boldsymbol{\nu}_i^\mathsf{T}\|_1 - m_{ji}$ over all $i \in \mathcal{K}_{\mathrm{Tx}}, j \in \mathcal{K}_{\mathrm{Rx}}$ provides the minimal feasible n. Altogether, the minimal necessary number of n dimensions is lower bounded by:

$$\begin{aligned} n &\geq \max_{j\in\mathcal{K}_{\mathrm{Rx}},\, i\in\mathcal{K}_{\mathrm{Tx}}} \left(\|\boldsymbol{\mu}_j\|_1 + \|\boldsymbol{\nu}_i^\mathsf{T}\|_1 - m_{ji}\right) \\ &= \max_{j\in\mathcal{K}_{\mathrm{Rx}},\, i\in\mathcal{K}_{\mathrm{Tx}}} \left(\textstyle\sum_{k=1}^{K_{\mathrm{Tx}}} m_{jk} + \sum_{l=1}^{K_{\mathrm{Rx}}} m_{li} - m_{ji}\right), \end{aligned} \tag{3.37}$$

with $j \in \mathcal{K}_{\mathrm{Rx}}$ and $i \in \mathcal{K}_{\mathrm{Tx}}$ for a fixed messaging matrix $\boldsymbol{M}$. Thus, from the definition in (2.20), the DoF of a $K_{\mathrm{Tx}} \times K_{\mathrm{Rx}}$ - user X- channel are upper bounded by:

$$\mathrm{DoF} \leq \frac{\sum_{j=1}^{K_{\mathrm{Rx}}} \sum_{i=1}^{K_{\mathrm{Tx}}} m_{ji}}{\max\limits_{j\in\mathcal{K}_{\mathrm{Rx}},\, i\in\mathcal{K}_{\mathrm{Tx}}} \left(\sum_{k=1}^{K_{\mathrm{Tx}}} m_{jk} + \sum_{l=1}^{K_{\mathrm{Rx}}} m_{li} - m_{ji}\right)}. \tag{3.38}$$

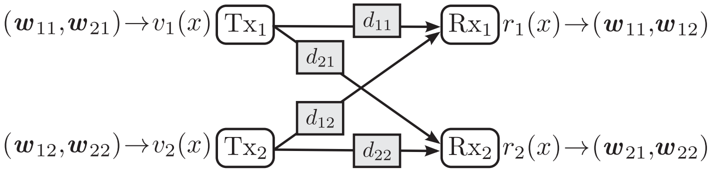


Figure 3.5: The polynomial 2 -user X- channel with messages W_{11}, W_{12}, W_{21} and W_{22} between Tx_1, Tx_2, Rx_1 and Rx_2 and the cyclic shifts d_{11}, d_{12}, d_{21} and d_{22} of the transfer matrix $\boldsymbol{D}$.

3.3 Cyclic IA on the Cyclic Polynomial 2 - User X - Channel

A 2 -user X- channel is physically equivalent to a 2IFC. However, it has a number of four independent messages $\boldsymbol{w}_{11}, \boldsymbol{w}_{21}, \boldsymbol{w}_{12}$ and $\boldsymbol{w}_{22}$ instead of only two independent messages $\boldsymbol{w}_{11}$ and $\boldsymbol{w}_{22}$ for two separate dedicated user-pairs. The sets of indices are again $\mathcal{K}_{\mathrm{Tx}} = \mathcal{K}_{\mathrm{Rx}} = \mathcal{K} = \{1, 2\}$. The 2 -user X- channel as depicted in Figure 3.5 does not only combine the properties of a 2 -user multiple-access, a 2 -user broadcast and a 2 -user interference channel, it also serves as the most basic example for the application of cyclic IA. The X- channel has also been considered in [11] and in [38] for the initial development of the IA principle for time-varying MIMO X- channels.

In terms of the CPCM, the task is to convey and decode four dedicated messages $\boldsymbol{w}_{11}$, $\boldsymbol{w}_{12}$, $\boldsymbol{w}_{21}$ and $\boldsymbol{w}_{22}$ interference-free within n dimensions. The scheme is optimal if n is minimal and still feasible. The channel transfer matrix of the X- channel is the same as in (3.16), namely:

$$\boldsymbol{D} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

For an introductory discussion of cyclic IA, we begin with a simplified messaging matrix $\boldsymbol{M} = \mathbf{1}_{2\times 2}$ with only a single submessage per user-pair. The general case with arbitrary message lengths will be treated afterwards in Sections 3.3.3 and 3.3.4. To simplify notation, and since $m_{ji} = 1$ for all $j, i \in \mathcal{K}$ holds, the superscripts $[0]$ are omitted, i. e., we use $W_{ji}^{[0]} = W_{ji} = \boldsymbol{w}_{ji}$ and $p_{ji}^{[0]} = p_{ji}$ here.

3.3.1 Perfect Cyclic Interference Alignment

We call interfering submessages to be *aligned* if they are received within the same dimension at an undesired receiver. In order to align the submessages W_{22} and W_{21} of $v_1(x)$ and $v_2(x)$, they must overlap in the same dimension at Rx_1, but they must remain distinct in different dimensions at Rx_2. To align the submessages W_{11} and W_{12} of $v_1(x)$ and $v_2(x)$, they must overlap at Rx_2, but remain distinct at Rx_1, accordingly. An example of such an alignment is shown in the table of Figure 3.6. The following

	x^0	x^1	x^2
$v_1(x)$	W_{11}	0	W_{21}
$v_2(x)$	0	W_{12}	W_{22}
$r_1(x)$	$W_{21}+W_{22}$	W_{11}	W_{12}
$r_2(x)$	$W_{11}+W_{12}$	W_{22}	W_{21}

Figure 3.6: Cyclic IA is applied on a 2-user X-channel with $n = 3$ dimensions. The following cyclic 'right-shifts' $d_{11} = d_{12} = x^1$, $d_{21} = x^3$ and $d_{22} = x^2$ are assumed. The parameters are $p_{11} = 0$, $p_{12} = 1$, $p_{21} = 2$, $p_{22} = 2$ and $n = 3$ and $\frac{4}{3}$ DoF are achieved. Aligned interference is highlighted in red.

polynomials with the offset parameters $p_{11}, p_{12}, p_{21}, p_{22} \in \mathbb{N}_0$ are used for transmission:

$$v_1(x) = W_{11}x^{p_{11}} + W_{21}x^{p_{21}}, \tag{3.39}$$
$$v_2(x) = W_{12}x^{p_{12}} + W_{22}x^{p_{22}}. \tag{3.40}$$

Then by (2.18), the received polynomials at Rx_1 and Rx_2 yield:

$$\begin{aligned} r_1(x) \equiv\ & d_{11}W_{11}x^{p_{11}} + d_{12}W_{12}x^{p_{12}} + \\ & d_{11}W_{21}x^{p_{21}} + d_{12}W_{22}x^{p_{22}} \mod(x^n - 1), \end{aligned} \tag{3.41}$$
$$\begin{aligned} r_2(x) \equiv\ & d_{21}W_{21}x^{p_{21}} + d_{22}W_{22}x^{p_{22}} + \\ & d_{21}W_{11}x^{p_{11}} + d_{22}W_{12}x^{p_{12}} \mod(x^n - 1). \end{aligned} \tag{3.42}$$

The messages dedicated for Rx_j must be linearly decodable. Thus, we demand that the *multiple-access interference conditions* hold with the indices $j \in \mathcal{K}$, and $i \neq l \in \mathcal{K}$:

$$d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{jl}} \mod(x^n - 1). \tag{3.43}$$

The two messages from Tx_i dedicated for different receivers Rx_j and Rx_k must also be linear decodable. Accordingly, we demand that the *intra-user interference conditions* with $j \neq k \in \mathcal{K}$, $i \in \mathcal{K}$, hold:

$$x^{p_{ji}} \not\equiv x^{p_{ki}} \mod(x^n - 1). \tag{3.44}$$

Moreover, the interfering messages must also be received separately from the dedicated messages at each Rx_j. Hence, the *inter-user interference conditions* must hold with $j \neq k \in \mathcal{K}$, $i \neq l \in \mathcal{K}$:

$$d_{ji}x^{p_{ji}} \not\equiv d_{jl}x^{p_{kl}} \mod(x^n - 1). \tag{3.45}$$

These negated congruences (3.43), (3.44) and (3.45) are the *separability conditions* of the cyclic polynomial 2-user X-channel. The separability conditions do not preclude to align intra- and inter-user interference to a single dimension at each Rx_j, which is expressed by:

$$d_{ji}x^{p_{ki}} \equiv d_{jl}x^{p_{kl}} \mod(x^n - 1), \tag{3.46}$$

with $j \neq k \in \mathcal{K}$ and $i \neq l \in \mathcal{K}$. Such an alignment is called *perfect* because the two different interference signals perfectly overlap in exactly the same dimensions per receiver.

We further remark that (3.46) substituted into (3.45) yields (3.44):

$$\begin{aligned} d_{ji}x^{p_{ji}} &\not\equiv d_{jl}x^{p_{kl}} \bmod(x^n - 1) \\ \Rightarrow d_{ji}x^{p_{ji}} &\not\equiv d_{ji}x^{p_{ki}} \bmod(x^n - 1) \\ \Rightarrow x^{p_{ji}} &\not\equiv x^{p_{ki}} \bmod(x^n - 1), \end{aligned}$$

with the indices $j \neq k \in \mathcal{K}$ and $i \neq l \in \mathcal{K}$. Hence, we can neglect the condition (3.45) if both (3.44) and (3.46) hold.

From the upper bound (3.38) and $\boldsymbol{M} = \mathbf{1}_{2\times 2}$, the 2 user X- channel is upper bounded by $\frac{4}{3}$ DoF.

Theorem 3.6. *A perfect cyclic IA scheme for the cyclic polynomial X- channel satisfying the separability conditions of the X-channel exists, if and only if the condition $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod(x^n - 1)$ and $n = 3$ hold. Then, cyclic IA achieves the upper bound of $\frac{4}{3}$ DoF.*

Proof:

(a) Necessity of $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod(x^n - 1)$, $n \in \mathbb{N}$:
By assuming the contraposition, $\mathsf{det}(\boldsymbol{D}) \equiv 0 \bmod(x^n - 1)$ yields (cf. (3.23)):

$$\begin{aligned} \mathsf{det}(\boldsymbol{D}) &\equiv 0 \bmod(x^n - 1) \\ \Rightarrow d_{11}d_{22} - d_{21}d_{12} &\equiv 0 \bmod(x^n - 1) \\ \Rightarrow d_{11}d_{22} &\equiv d_{21}d_{12} \bmod(x^n - 1) \\ \Rightarrow d_{ji}d_{kl} &\equiv d_{ki}d_{jl} \bmod(x^n - 1), \end{aligned} \tag{3.47}$$

with the indices $j \neq k \in \mathcal{K}$ and $i \neq l \in \mathcal{K}$. Including (3.47) into condition (3.46) yields:

$$\begin{aligned} d_{ji}x^{p_{ki}} &\equiv d_{jl}x^{p_{kl}} \bmod(x^n - 1) \\ \Rightarrow d_{ki}x^{p_{ki}} &\equiv d_{kl}x^{p_{kl}} \bmod(x^n - 1). \end{aligned}$$

Relabeling the indices $j \leftrightarrow k$ provides:

$$\Rightarrow d_{ji}x^{p_{ji}} \equiv d_{jl}x^{p_{jl}} \bmod(x^n - 1),$$

and contradicts (3.43) for any $n \in \mathbb{N}$.

(b) Necessity of $n > 2$ dimensions:
Assume $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod(x^n - 1)$ holds. We consider the right-hand sides of (3.43) and (3.44). (3.44) is expanded by d_{ji}.

$$\begin{aligned} (3.43):\ d_{ji}x^{p_{ji}} &\not\equiv d_{jl}x^{p_{jl}} \bmod(x^n - 1), \\ (3.44):\ d_{ji}x^{p_{ji}} &\not\equiv d_{ji}x^{p_{ki}} \bmod(x^n - 1). \end{aligned}$$

These right-hand side terms must also be pairwise distinct, since we can relabel the indices $i \leftrightarrow l$ in (3.48) to obtain (3.45):

$$\begin{aligned} d_{jl}x^{p_{jl}} &\not\equiv d_{ji}x^{p_{ki}} \bmod(x^n - 1) \\ \Leftrightarrow d_{ji}x^{p_{ji}} &\not\equiv d_{jl}x^{p_{kl}} \bmod(x^n - 1). \end{aligned} \tag{3.48}$$

There is no solution to satisfy all three conditions on $d_{ji}x^{p_{ji}}$ with only $n = 1$ or $n = 2$ dimensions.

(c) Sufficiency of $n = 3$ *and* $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod (x^n - 1)$:
From the perfect IA condition in (3.46), the following holds:

$$x^{p_{12}} \equiv d_{22}^{-1} d_{21} x^{p_{11}} \bmod (x^3 - 1), \tag{3.49}$$
$$x^{p_{21}} \equiv d_{11}^{-1} d_{12} x^{p_{22}} \bmod (x^3 - 1). \tag{3.50}$$

Furthermore, the condition (3.44) must hold:

$$x^{p_{11}} \not\equiv x^{p_{21}} \bmod (x^3 - 1), \tag{3.51}$$
$$x^{p_{12}} \not\equiv x^{p_{22}} \bmod (x^3 - 1). \tag{3.52}$$

The insertion of (3.49) and (3.50) into condition (3.51) yields:

$$x^{p_{22}} \not\equiv d_{22} d_{11} d_{21}^{-1} d_{12}^{-1} x^{p_{12}} \bmod (x^3 - 1), \tag{3.53}$$

Due to the determinant in (3.47), the following holds:

$$d_{12}^{-1} d_{21}^{-1} d_{22} d_{11} \not\equiv 1 \bmod (x^3 - 1).$$

W.l.o.g., we can fix p_{11} and compute p_{12} using (3.49). We can determine a solution for p_{22} from (3.52) and (3.53) only if $n > 2$. For $n = 3$ the solution of p_{22} is unique. The remaining parameter p_{21} is derived using (3.52).

The validity of condition (3.43) is yet to check. Inserting (3.49) and (3.50) into (3.43) for all cases provides:

$$x^{p_{11}} \not\equiv d_{12} d_{21} d_{22}^{-1} d_{11}^{-1} x^{p_{11}} \bmod (x^3 - 1),$$
$$x^{p_{21}} \not\equiv d_{22} d_{11} d_{12}^{-1} d_{21}^{-1} x^{p_{21}} \bmod (x^3 - 1).$$

Both conditions are satisfied by prerequisite, since (3.47) holds. Altogether, there is a solution for cyclic IA on the X-channel with $n = 3$ dimensions and $M = 4$ messages satisfying the separability conditions and achieving the upper bound of $\frac{4}{3}$ DoF. ■

Note that this theorem also includes the example of the 2-user X-channel considered in [37, Figure 1] for $d_{11} = d_{21} = x^1$, $d_{12} = x^0$, $d_{22} = x^2$ and $p_{21} = p_{12} = 0$, $p_{11} = p_{22} = 1$.

Corollary 3.7. *If* $\mathsf{det}(\boldsymbol{D}) \equiv 0 \bmod (x^n - 1)$ *holds, only* 1 *DoF can be achieved on the cyclic polynomial* 2*-user* X*-channel.*

Proof:
Theorem 3.6 (a) yields that interference signals can not be aligned if the condition $\mathsf{det}(\boldsymbol{D}) \equiv 0 \bmod (x^n - 1)$ holds. Each message must be received distinctly within its own dedicated offset, i. e., a MA scheme demands $n \geq 4$. As in Lemma 3.3, only 1 DoF is achievable by decoding all four messages at each receiver. ■

Note that, if $\mathsf{det}(\boldsymbol{D}) \equiv 0 \bmod (x^n - 1)$ holds, the cyclic polynomial X-channel does not provide the *relativity of IA*. In other words, the influence of each subchannel to the receivers does not provide a sufficiently diverse view of the transmitted signals, so that the observed signals from both transmitters behave identically and the benefits of cyclic IA are precluded. An according exceptional case is also observed in the symmetric 2-user X-channel in terms of the LDCM in [21, Theorems 3.1 & 3.2] at full symmetry.

Before we devise a more generalized cyclic IA scheme for the X-channel with an arbitrary number of submessages, we first elaborate an inherent symmetry property of IA. The observed property is a main tool for the derivation of the capacity-achieving scheme.

3.3.2 Complementary Reciprocal Symmetry of Cyclic IA

Interestingly, aligning two messages from different transmitters at one receiver provides an inherent symmetry property in the given X-channel, if the necessary determinant condition $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod (x^n-1)$ holds. To elaborate this, we consider two parameters $p_1, p_2 \in \mathbb{N}$ for messages $\tilde{W}_1$, $\tilde{W}_2$ from Tx_1 and Tx_2, respectively. Let p_1 and p_2 be chosen such that these messages are aligned at Rx_2 at a fixed offset $\lambda_2 \in \mathbb{N}$:

$$d_{21}x^{p_1} \equiv d_{22}x^{p_2} \equiv x^{\lambda_2} \mod (x^n - 1). \tag{3.54}$$

This alignment results in the following received signals (denoted with tilde):

$$\tilde{r}_1(x) \equiv x^{\lambda_2}(d_{11}d_{21}^{-1}\tilde{W}_1 + d_{12}d_{22}^{-1}\tilde{W}_2) \mod (x^n - 1), \tag{3.55}$$

$$\tilde{r}_2(x) \equiv x^{\lambda_2}(\tilde{W}_1 + \tilde{W}_2) \mod (x^n - 1). \tag{3.56}$$

Conversely, we may choose p_1, p_2 such that two messages $\mathring{W}_1$, $\mathring{W}_2$ align at Rx_1 at a fixed offset $\lambda_1 \in \mathbb{N}$ instead:

$$d_{11}x^{p_1} \equiv d_{12}x^{p_2} \equiv x^{\lambda_1} \mod (x^n - 1). \tag{3.57}$$

The received signals of the second case yield (denoted with circle):

$$\mathring{r}_1(x) \equiv x^{\lambda_1}(\mathring{W}_1 + \mathring{W}_2) \mod (x^n - 1), \tag{3.58}$$

$$\mathring{r}_2(x) \equiv x^{\lambda_1}(d_{11}^{-1}d_{21}\mathring{W}_1 + d_{12}^{-1}d_{22}\mathring{W}_2) \mod (x^n - 1). \tag{3.59}$$

By comparing (3.55) with (3.59) and (3.56) with (3.58), we observe that the offsets of the polynomials $\tilde{r}_i(x)$ and $\mathring{r}_j(x)$ can be mutually converted by the following transformation[3]:

$$\mathring{r}_j(x) \equiv x^{\lambda_i+\lambda_j}\, \tilde{r}_i(x^{-1})\big|_{\tilde{W}_1 \to \mathring{W}_1, \tilde{W}_2 \to \mathring{W}_2} \mod (x^n - 1), \tag{3.60}$$

$$\tilde{r}_j(x) \equiv x^{\lambda_i+\lambda_j}\, \mathring{r}_i(x^{-1})\big|_{\mathring{W}_1 \to \tilde{W}_1, \mathring{W}_2 \to \tilde{W}_2} \mod (x^n - 1), \tag{3.61}$$

for $i \neq j \in \mathcal{K}$. The coefficients with the messages are substituted correspondingly. We call these two alignments *complementary reciprocal symmetric*[4], since the dedicated

[3] For the *reciprocal polynomial* of $p(x)$, the sequence of coefficients is reversed, i. e., we obtain $p(x^{-1}) = \sum_{k=0}^{n-1} p^{[k]}x^{-k}$.

[4] This property should not be confused with the related concept of the *reciprocity of alignment* in [40], where IA on the reciprocal channel with swapped transmitter-receiver pairs is considered instead.

signals and the aligned interference signals are swapped at the receivers and the offsets are reciprocally inverted along λ_i and shifted to λ_j (or vice versa). In other words, the complementary reciprocal symmetry basically states that aligning signals from both transmitters at Rx_1 provides a particular interference pattern at Rx_2, while vice versa, aligning two signals at Rx_2 provides the reciprocal interference pattern of the one observed before at Rx_1.

3.3.3 X- Channel with Generalized Message Lengths

Now, we generalize the perfect IA scheme of Section 3.3.1 such that the messages $\boldsymbol{w}_{ji}$ may have different lengths $m_{ji} \geq 1$. In contrast to the simplified case treated above, a scheme purely based on IA is not sufficient to achieve the upper bound. Similar to the 2IFC case, we also need to employ a LEaD scheme. Furthermore, we apply the complementary reciprocal symmetry property elaborated in the previous section.

The generalized messaging matrix of the 2-user X- channel is:

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \tag{3.62}$$

The total number of transmitted submessages is:

$$M = m_{11} + m_{21} + m_{12} + m_{22}. \tag{3.63}$$

In analogy to the MAC, BC and 2IFC, we use the vectors:

$$\boldsymbol{w}_{ji} = (W_{ji}^{[0]}, ..., W_{ji}^{[m_{ji}-1]}), \tag{3.64}$$

$$\boldsymbol{p}_{ji} = (p_{ji}^{[0]}, ..., p_{ji}^{[m_{ji}-1]}). \tag{3.65}$$

The transmitted polynomial from Tx_i is:

$$v_i(x) = \sum_{j=1}^{2} \sum_{k=1}^{m_{ji}} W_{ji}^{[k]} x^{p_{ji}^{[k]}}. \tag{3.66}$$

The received polynomials derived from (2.18) yield:

$$r_1(x) \equiv d_{11}v_1(x) + d_{12}v_2(x) \mod (x^n - 1),$$
$$r_2(x) \equiv d_{21}v_1(x) + d_{21}v_2(x) \mod (x^n - 1).$$

The separability conditions defined in Section 3.3.1 are now generalized for generalized message lengths, i.e., each condition also applies to any pair of submessages as follows:

Multiple-access interference conditions with
$i, j, l \in \mathcal{K}$, $i \neq l$, $t \in \{0, ..., m_{ji} - 1\}$, $t' \in \{0, ..., m_{jl} - 1\}$:

$$d_{ji}x^{p_{ji}^{[t]}} \not\equiv d_{jl}x^{p_{jl}^{[t']}} \mod (x^n - 1). \tag{3.67}$$

Intra-user interference conditions with
$i, j, k \in \mathcal{K}$, $j \neq k$, $t \in \{0, ..., m_{ji} - 1\}$, $t' \in \{0, ..., m_{ki} - 1\}$:

$$x^{p_{ji}^{[t]}} \not\equiv x^{p_{ki}^{[t']}} \mod (x^n - 1). \tag{3.68}$$

Inter-user interference conditions with
$i, j, k \in \mathcal{K}$, $i \neq l$, $k \neq j$, $t \in \{0, ..., m_{ji} - 1\}$, $t' \in \{0, ..., m_{ki} - 1\}$:

$$d_{ji} x^{p_{ji}^{[t]}} \not\equiv d_{jl} x^{p_{ki}^{[t']}} \mod (x^n - 1). \tag{3.69}$$

3.3.4 Achievability of Perfect Cyclic IA with Linear Coding

Despite the vast amount of separability conditions for the X- channel as given above, a communication scheme that includes cyclic IA, LEaD and exploits the property of complementary reciprocal symmetry of IA, can achieve the corresponding upper bounds and the maximal sum-rate of up to $\frac{4}{3}$ DoF for equal message lengths. This is shown in the following theorem.

Theorem 3.8. *A combined cyclic IA and LEaD scheme achieves the upper bound:*

$$\mathrm{DoF} \leq \frac{m_{11} + m_{21} + m_{12} + m_{22}}{\max_{i \neq j \in \mathcal{K}} (m_{ii} + m_{ij} + \max(m_{ji}, m_{jj}))},$$

on the 2 -user X- channel with arbitrary message lengths, if the total number of dimensions is $n = \max_{i \neq j \in \mathcal{K}} (m_{ii} + m_{ij} + \max(m_{ji}, m_{jj}))$, and if $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod(x^n - 1)$ holds.

Proof:
(a) Necessity of $n \geq \max_{i \neq j \in \mathcal{K}} (m_{ii} + m_{ij} + \max(m_{ji}, m_{jj}))$:
This upper bound on n is a special case of the upper bound already shown in (3.38) for the given $\boldsymbol{M}$ in (3.62).

(b) Necessity of $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod (x^n - 1)$:
For $\mathsf{det}(\boldsymbol{D}) \equiv 0 \mod(x^n - 1)$, cyclic IA contradicts to the separability conditions as already proven in Theorem 3.6 (a).

(c) Sufficiency of cyclic IA for $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod (x^n - 1)$ and the given lower bound on n:
Our cyclic IA scheme with LEaD is outlined as follows:

(i) *Align the primary interference space at* Rx_2:
Interference signals are pairwise aligned at Rx_2 beginning at offset λ_2 to form a primary interference space confined to a continuous frame[5].

(ii) *Linear encoding and decoding extension at* Rx_2:
The LEaD extension ensures that dedicated signals are decodable at Rx_1. Furthermore, this extension provides a frame of sufficiently many unused dimensions at Rx_1 such that the next step (iii) can be applied.

(iii) *Complementary reciprocal symmetric alignment at* Rx_1:
Using the complementary reciprocal symmetry of cyclic IA, the remaining signals are reciprocally aligned at Rx_1 such that the already aligned dedicated signals still remain decodable.

[5]We call a continuous sequence of offsets a *frame*.

First, let $m_{11} = m_{12} = m_{21} = m_{22} = k$, with $k \in \mathbb{N}$, so that lower bound on n yields $n = 3k$ with a maximal number of dedicated submessages k. Clearly, it suffices to prove three cases: $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$.

Step (i) - Alignment of the Primary Interference Space:

In this initial step, k parameters $p_{11}^{[0]}, \ldots, p_{11}^{[k-1]}$ and k parameters $p_{12}^{[0]}, \ldots, p_{12}^{[k-1]}$ for the messages $\boldsymbol{w}_{11}$ and $\boldsymbol{w}_{12}$ are fixed such that the interfering submessages $W_{11}^{[l]}$ and $W_{12}^{[l]}$ are pairwise aligned at Rx_2 for $l = 0, \ldots, k-1$ in a frame of k distinct dimensions. We choose the offset of the first aligned submessages $W_{11}^{[0]}$, $W_{12}^{[0]}$ at Rx_2 as $\lambda_2 \in \{0, \ldots, n-1\}$, so that:

$$d_{21} x^{p_{11}^{[l]}} \equiv d_{22} x^{p_{12}^{[l]}} \equiv x^{\lambda_2 + l} \mod (x^n - 1). \tag{3.70}$$

So far, the transmitted signals from Tx_i, for $i \in \mathcal{K}$, are:

$$\tilde{v}_i(x) = \sum_{l=0}^{k-1} W_{1i}^{[l]} x^{p_{1i}^{[l]}}. \tag{3.71}$$

The following signals are received at Rx_1 and Rx_2:

$$\tilde{r}_1(x) \equiv \sum_{l=0}^{k-1} x^{\lambda_2 + l} (d_{11}^{-1} d_{21} W_{11}^{[l]} + d_{12}^{-1} d_{22} W_{12}^{[l]}) \mod (x^n - 1), \tag{3.72}$$

$$\tilde{r}_2(x) \equiv \sum_{l=0}^{k-1} x^{\lambda_2 + l} (W_{11}^{[l]} + W_{12}^{[l]}) \mod (x^n - 1) \tag{3.73}$$

This allocation ensures that the aligned interference at Rx_2 is confined to a frame of k dimensions at offsets $x^{\lambda_2}, \ldots, x^{\lambda_2 + k - 1}$. We call this frame of aligned interference the *primary interference space* $\mathcal{I}_2 = \mathcal{I}_{21} \cup \mathcal{I}_{22}$ at Rx_2.

Let a frame of $\tau \in \mathbb{N}$ dedicated submessages transmitted from Tx_i and received at Rx_j be denoted by the set:

$$\mathcal{S}_{ji}^{[\tau]} = \{d_{ji} W_{ji}^{[0]} x^{p_{ji}^{[0]}}, \ldots, d_{ji} W_{ji}^{[\tau-1]} x^{p_{ji}^{[\tau-1]}}\}. \tag{3.74}$$

Currently, two different frames, each of k dedicated submessages, occupy the signal space at Rx_1 as illustrated by the white areas of Figure 3.7. Let the parameters τ_{11} and τ_{12} denote the offset of the first submessage for the frames of $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$ as received at Rx_1, respectively:

$$\tau_{11} \equiv \lambda_2 - \delta_{21} + \delta_{11} \pmod{n}, \tag{3.75}$$

$$\tau_{12} \equiv \lambda_2 - \delta_{22} + \delta_{12} \pmod{n}. \tag{3.76}$$

The offset which is next to the end of such a k-dimensional frame is at:

$$\tau_1 = \tau_{11} + k, \tag{3.77}$$

$$\tau_2 = \tau_{12} + k. \tag{3.78}$$

We deliberately avoid the $(\bmod\, n)$-operation in (3.77) and (3.78) to maintain a linear ordering relationship over a cyclically unrolled period. In order to determine the size of the yet unused offsets at Rx_1 (grey shading), we need to compute the distance Δ_1

between τ_1 and τ_{12}, and also the distance Δ_2 between τ_2 and τ_{11} (cf. clock-wise distance metric in [60]):

$$\Delta_1 = \begin{cases} \tau_{12} - \tau_1, & \text{if } \tau_{11} < \tau_{12}, \\ n + \tau_{12} - \tau_1, & \text{if } \tau_{11} > \tau_{12}, \end{cases} \tag{3.79}$$

$$\Delta_2 = \begin{cases} n + \tau_{11} - \tau_2, & \text{if } \tau_{11} < \tau_{12}, \\ \tau_{11} - \tau_2, & \text{if } \tau_{11} > \tau_{12}. \end{cases} \tag{3.80}$$

If Δ_1 or Δ_2 yields a negative value, then the frames $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$ overlap. An example of the parameters τ_{11}, τ_1, τ_{12}, τ_2, and distances Δ_1, Δ_2 is illustrated in Figure 3.7.

Case (a): If there is *no intersection* at Rx_1, i. e., if $\mathcal{S}_{11} \cap \mathcal{S}_{12} = \varnothing$, then we will obtain two unused frames of non-negative length $0 \le \Delta_1 < k$ and $0 \le \Delta_2 < k$, respectively. We define the following auxiliary parameters for the maximal and the minimal distance of Δ_1 and Δ_2:

$$\underline{\Delta}_3 = \min(\Delta_1, \Delta_2), \tag{3.81}$$

$$\overline{\Delta}_3 = \max(\Delta_1, \Delta_2). \tag{3.82}$$

In this non-intersecting case, $|\mathcal{S}_{11} \cup \mathcal{S}_{12}| = 2k$ dimensions are occupied by two disjoint frames of k dimensions each. A number of $n - 2k = k$ dimensions at Rx_1 remains unused yet. The unused offsets cover exactly $\underline{\Delta}_3 + \overline{\Delta}_3 = \Delta_1 + \Delta_2 = n - 2k = k$ dimensions in this case (cf. Figure 3.7(a)). Both parameters $\underline{\Delta}_3$ and $\overline{\Delta}_3$ are non-negative integers here.

Case (b): Otherwise, the dedicated signals must intersect at Rx_1, i. e., $\mathcal{S}_{11} \cap \mathcal{S}_{12} \neq \varnothing$. We obtain that either Δ_1 or Δ_2 is negative. $\underline{\Delta}_3$ also becomes negative. The absolute value $|\underline{\Delta}_3|$ indicates the number of overlapping dimensions. $2k - |\underline{\Delta}_3|$ dimensions are occupied by (partially overlapping) dedicated signals. A number of $n - \overline{\Delta}_3$ dimensions at Rx_1 remains unused yet. The fully overlapping case $\mathcal{S}_{11} = \mathcal{S}_{12}$ is already excluded by $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod (x^n - 1)$.

Step (ii) - Linear Encoding and Decoding (LEaD) Extension:

We propose a LEaD scheme in order to provide the following three properties for the received signals at Rx_1 and Rx_2:

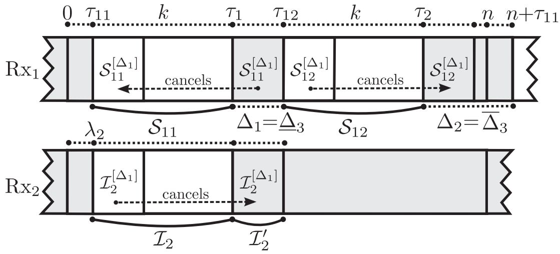
I All dedicated signals at Rx_1 are linearly decodable.

II A frame of at least k dimensions is available for the yet unallocated primary interference space $\mathcal{I}_1$ at Rx_1.

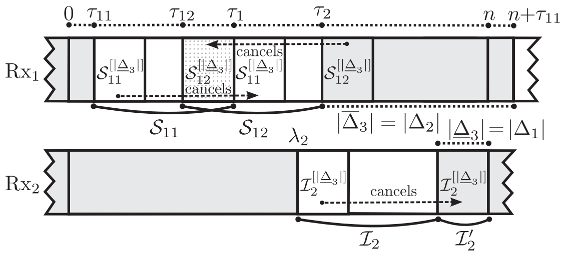
III A frame of $2k$ unused dimensions is available at Rx_2. This frame is retained for the remaining dedicated messages $\boldsymbol{w}_{21}$ and $\boldsymbol{w}_{22}$.

We repeat the first $|\underline{\Delta}_3|$ submessages $W_{11}^{[l]}, W_{12}^{[l]}$, for $l = 0, \ldots, |\underline{\Delta}_3| - 1$ at Rx_2, by aligning them next to $\mathcal{I}_2$:

$$d_{21} x^{q_{11}^{[l]}} \equiv d_{22} x^{q_{12}^{[l]}} \equiv x^{\lambda_2 + k + l} \mod (x^n - 1), \tag{3.83}$$



(a) Non-overlapping frames in n dimensions: $\mathcal{S}_{11} \cap \mathcal{S}_{12} = \varnothing$.



(b) Overlapping frames in n dimensions: $\mathcal{S}_{11} \cap \mathcal{S}_{12} \neq \varnothing$.

Figure 3.7: The illustration shows an example for the case of non-overlapping frames $\mathcal{S}_{11} \cap \mathcal{S}_{12} = \varnothing$ at the top, and for the case of overlapping frames $\mathcal{S}_{11} \cap \mathcal{S}_{12} \neq \varnothing$ at the bottom, as allocated by step (i) at Rx$_1$. Yet unused dimensions are shown in grey. For step (ii), the linear encoding and decoding extension is depicted for $\tau_{11} < \tau_{12}$, and $0 < \Delta_1 < \Delta_2$. The cancellation of redundant or overlapping signals is indicated by dashed arrows. Overlapping offsets are dotted in grey.

with the additional offset parameters $q_{1i}^{[l]}$ for the repeated submessages $W_{1i}^{[l]}$. The extended transmission signals from Tx$_i$, for $i \in \mathcal{K}$, become:

$$\tilde{v}_i(x) = \textstyle\sum_{l=0}^{k-1} W_{1i}^{[l]} x^{p_{1i}^{[l]}} + \sum_{l'=0}^{|\underline{\Delta}_3|-1} W_{1i}^{[l']} x^{q_{1i}^{[l']}}, \tag{3.84}$$

and the corresponding received signals are:

$$\tilde{r}_1(x) \equiv \textstyle\sum_{l=0}^{k-1} x^{\lambda_2+l}(d_{11}^{-1}d_{21}W_{11}^{[l]} + d_{12}^{-1}d_{22}W_{12}^{[l]}) + \tag{3.85}$$

$$\textstyle\sum_{l'=0}^{|\underline{\Delta}_3|-1} x^{\lambda_2+k+l'}(d_{11}^{-1}d_{21}W_{11}^{[l']} + d_{12}^{-1}d_{22}W_{12}^{[l']}) \mod (x^n - 1), \tag{3.86}$$

$$\tilde{r}_2(x) \equiv \textstyle\sum_{l=0}^{k-1} x^{\lambda_2+l}(W_{11}^{[l]} + W_{12}^{[l]}) + \sum_{l'=0}^{|\underline{\Delta}_3|-1} x^{\lambda_2+k+l'}(W_{11}^{[l']} + W_{12}^{[l']}) \mod (x^n - 1). \tag{3.87}$$

This extended alignment scheme produces a *secondary interference space* $\mathcal{I}_2'$ at Rx$_2$ with $|\underline{\Delta}_3|$ submessages. The secondary interference space $\mathcal{I}_2'$ at Rx$_2$ is cancelled by using the first $0, ..., |\underline{\Delta}_3| - 1$ dimensions of the primary interference space $\mathcal{I}_2$. This cancellation procedure is depicted by the dashed arrows in the lower graphs of Figure 3.7. Thus, the effective interference space left over at Rx_2 after cancellation is again $\mathcal{I}_2$, with only k dimensions.

In the case (a) with $\mathcal{S}_{11} \cap \mathcal{S}_{12} = \varnothing$ at Rx_1, the smaller gap of $|\underline{\Delta}_3|$ unused dimensions is filled up with the corresponding repeated dedicated signals $\mathcal{S}_{ij}^{[|\underline{\Delta}_3|]}$. This repetition in the smaller gap is used to cancel out the corresponding dedicated signals as depicted in Figure 3.7(a). In the larger gap with $|\overline{\Delta}_3|$ dimensions, the additional repetition of signals are cancelled by using the corresponding dedicated signals.

In the contrary case (b), we have $\mathcal{S}_{11} \cap \mathcal{S}_{12} \neq \varnothing$ at Rx_1. Now, a number of $|\underline{\Delta}_3|$ dimension overlaps at Rx_1. Likewise, we extend the signal as in (3.83) with the first $|\underline{\Delta}_3|$ submessages. At Rx_1, the repeated signals $\mathcal{S}_{1j}^{[|\underline{\Delta}_3|]}$ occur at τ_j, correspondingly. The overlapping parts are cancelled using the corresponding interference-free signals. This extension and cancellation procedure for this second case is depicted in Figure 3.7(b).

In both cases, the dedicated signals and the repeated signals form a continuous frame of $2k$ dimensions at Rx_1 after performing the cancellation scheme as described above. The dedicated signals allocated so far are decodable. Moreover, a frame of k dimensions is yet left unused after cancellation at Rx_1 and a frame of $2k$ dimensions is left unused at Rx_2. Hence, the LEaD-extension provides all three desired properties I, II, and III.

Step (iii) - Complementary Reciprocal Symmetric Alignment:

In this step, we derive the complementary alignment of interfering signals at Rx_1. We align the interference at Rx_1, caused by $W_{21}^{[l]}$ and $W_{22}^{[l]}$, at an offset $\lambda_1 \in \{0, ..., n-1\}$ by:

$$d_{11} x^{p_{21}^{[l]}} \equiv d_{12} x^{p_{22}^{[l]}} \equiv x^{\lambda_1 - l} \mod (x^n - 1), \tag{3.88}$$

for $l = 0, ..., k-1$. This primary interference space $\mathcal{I}_1$ is aligned in the reciprocal direction, when compared to the IA at Rx_2 in (3.70). The offset λ_1 is uniquely chosen by:

$$\lambda_1 = \begin{cases} \tau_{11} - 1 + [\Delta_1]^+, & \text{if } \Delta_1 \leq \Delta_2, \\ \tau_{12} - 1 + [\Delta_2]^+, & \text{if } \Delta_2 < \Delta_1, \end{cases} \tag{3.89}$$

so that $\mathcal{I}_1$ is located exactly within the frame of k yet unused (or cancelled) offsets.

Analogously to step (ii), we also apply a LEaD-extension to provide decodability for the dedicated signals $W_{21}^{[l]}$ and $W_{22}^{[l]}$. Again, the first $|\underline{\Delta}_3|$ submessages $W_{21}^{[l]}, W_{22}^{[l]}$, with $l = 0, ..., |\underline{\Delta}_3| - 1$, are repeated next to the primary interference space $\mathcal{I}_1$, also in the reciprocal direction, so that they are aligned at:

$$d_{11} x^{q_{21}^{[l]}} \equiv d_{12} x^{q_{22}^{[l]}} \equiv x^{\lambda_1 - k - l} \mod (x^n - 1), \tag{3.90}$$

with the offsets $q_{2i}^{[l]}$ for the repeated submessages $W_{2i}^{[l]}$, for $i \in \mathcal{K}$. The linearly extended transmission signal $\mathring{v}_i(x)$ from Tx_i is:

$$\mathring{v}_i(x) = \textstyle\sum_{l=0}^{k-1} W_{2i}^{[l]} x^{p_{2i}^{[l]}} + \sum_{l'=0}^{|\underline{\Delta}_3|-1} W_{2i}^{[l']} x^{q_{2i}^{[l']}}. \tag{3.91}$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6
$v_1(x)$	$W_{11}^{[2]}+W_{21}^{[0]}$	$W_{21}^{[1]}$	$W_{21}^{[0]}$	$W_{11}^{[0]}$	$W_{11}^{[1]}$	$W_{11}^{[0]}+W_{11}^{[2]}$	0
$v_2(x)$	0	$W_{12}^{[0]}$	$W_{12}^{[1]}$	$W_{12}^{[0]}$	$W_{22}^{[0]}$	$W_{22}^{[1]}$	$W_{22}^{[0]}$
$r_1(x)$	$W_{11}^{[2]}+\cancel{W_{21}^{[0]}+W_{22}^{[0]}}$	$W_{22}^{[1]}+W_{21}^{[1]}$	$W_{21}^{[0]}+W_{22}^{[0]}$	$W_{11}^{[0]}$	$W_{11}^{[1]}+\cancel{W_{12}^{[0]}}$	$W_{12}^{[1]}+\cancel{W_{11}^{[0]}}+\cancel{W_{11}^{[2]}}$	$W_{12}^{[0]}$
$r_2(x)$	$W_{11}^{[0]}+W_{12}^{[0]}$	$W_{11}^{[1]}+W_{12}^{[1]}$	$\cancel{W_{11}^{[0]}+W_{12}^{[0]}}+W_{11}^{[2]}$	$W_{22}^{[0]}$	$W_{22}^{[1]}+\cancel{W_{21}^{[0]}}+\cancel{W_{11}^{[2]}}$	$W_{21}^{[1]}+\cancel{W_{22}^{[0]}}$	$W_{21}^{[0]}$

Figure 3.8: An example of cyclic IA with the LEaD-extension for the 2 -user X- channel with $n = 7 = 3{\cdot}2{+}1$ dimensions ($\Rightarrow k = 2$). Let the channel matrix $\boldsymbol{D}$ impose the following cyclic 'right'-shifts $d_{11} = x^0$, $d_{12} = x^3$, $d_{21} = x^4$, $d_{22} = x^6$ and let $m_{11} = 3$, $m_{12} = m_{21} = m_{22} = 2$. The parameters from the scheme are $\lambda_2 = 0$, $\lambda_1 = 2$, $\tau_{11} = 3$, $\tau_{12} = 4$, $\tau_1 = 5$, $\tau_2 = 6$, $\Delta_1 = -1$, and $\Delta_2 = 4$. The primary interference spaces are highlighted in red and the secondary interference spaces in green. The complementary reciprocal symmetry is satisfied, when ignoring $W_{11}^{[2]}$, as highlighted in blue. The cancellation of interference by the linear decoder is indicated by the terms crossed out. Altogether, all $M = 9$ dedicated submessages are decodable within 7 dimensions achieving the proposed upper bound.

This provides the following received signals at Rx_1 and Rx_2:

$$\mathring{r}_1(x) \equiv \sum_{l=0}^{k-1} x^{\lambda_1 - l}(W_{21}^{[l]} + W_{22}^{[l]}) + \sum_{l'=0}^{|\Delta_3|-1} x^{\lambda_1-k-l'}(W_{21}^{[l']} + W_{22}^{[l']}) \mod (x^n - 1), \quad (3.92)$$

$$\mathring{r}_2(x) \equiv \sum_{l=0}^{k-1} x^{\lambda_1 - l}(d_{11}d_{21}^{-1}W_{21}^{[l]} + d_{12}d_{22}^{-1}W_{22}^{[l]}) + \quad (3.93)$$

$$\sum_{l'=0}^{|\Delta_3|-1} x^{\lambda_1-k-l'}(d_{11}d_{21}^{-1}W_{21}^{[l']} + d_{12}d_{22}^{-1}W_{22}^{[l']}) \mod (x^n - 1). \quad (3.94)$$

Now, we combine the transmitted signals (3.84) and (3.91) of Tx_i:

$$v_i(x) = \tilde{v}_i(x) + \mathring{v}_i(x). \quad (3.95)$$

At each Rx_j, the combined received signals yield from $\tilde{r}_i(x)$ and $\mathring{r}_i(x)$, respectively:

$$r_j(x) \equiv \tilde{r}_j(x) + \mathring{r}_j(x) \mod (x^n - 1). \quad (3.96)$$

For $r_1(x)$ all dedicated signals are linearly decoded as shown in step (ii). Recall that the primary interference space $\mathcal{I}_1$ fits perfectly into the formerly unused (or cancelled) signal space at Rx_1. The secondary interference space $\mathcal{I}_1'$ is cancelled using $\mathcal{I}_1$. The proposed alignment scheme satisfies the following complementary reciprocal symmetry relationship for $r_2(x)$ (cf. (3.60), (3.61)) by construction:

$$\mathring{r}_2(x) \equiv x^{\lambda_1+\lambda_2}\, \tilde{r}_1(x^{-1})\big|_{\boldsymbol{w}_{11}\to\boldsymbol{w}_{21},\, \boldsymbol{w}_{12}\to\boldsymbol{w}_{22}} \mod (x^n - 1), \quad (3.97)$$

$$\tilde{r}_2(x) \equiv x^{\lambda_1+\lambda_2}\, \mathring{r}_1(x^{-1})\big|_{\boldsymbol{w}_{21}\to\boldsymbol{w}_{11},\, \boldsymbol{w}_{22}\to\boldsymbol{w}_{12}} \mod (x^n - 1). \quad (3.98)$$

The resulting $r_2(x) = \tilde{r}_2(x) + \mathring{r}_2(x)$ is complementary reciprocal symmetric w.r.t. $r_1(x)$. Due to the corresponding relabelling of submessages, all dedicated signals for Rx_2 are decodable now, as well. At this point, the case $n = 3k$, for $k \in \mathbb{N}$, is already proven.

For the case that the minimal necessary n is lower bounded by $n = 3k + 1$, we may consider $m_{11} = k + 1$, $m_{12} = m_{21} = m_{22} = k$, w.l.o.g. As in the previous scheme, k pairs of submessages are aligned in the same way as in steps (i) to (iii) at both receivers. But now, the lower bound on n is limited by $3k + 1$ instead of only $3k$ dimensions. Thus, there is still one unused offset left over at both receivers after the cancellation of known interference. Tx_1 allocates its remaining $k + 1$-th submessage $W_{11}^{[k]}$ in $v_1(x)$ at those two offsets, such that one $W_{11}^{[k]}$ is received within the unused dimensions of the receivers Rx_1 and Rx_2 each. Both receivers will receive one interference-free $W_{11}^{[k]}$ and one interfering $W_{11}^{[k]}$ in general. Clearly, linear decoders can each cancel the interference at both receivers. Case $n = 3k + 2$ is treated analogously and hence it is omitted here.

An example of the proposed scheme for case $n = 3k + 1$ is provided in the table of Figure 3.8. As a result, all $m_{11} + m_{21} + m_{12} + m_{22}$ submessages are successfully conveyed within a total number of $n = \max_{i \neq j \in \mathcal{K}}(m_{ii} + m_{ij} + \max(m_{ji}, m_{jj}))$ dimensions for all cases. ■

The optimal choice for the number of submessages is $m_{ji} = k$ for a non-zero constant $k \in \mathbb{N}$, so that interference is perfectly aligned and the DoF are maximized by $\frac{4}{3}$.

3.4 Cyclic IA on the Cyclic Polynomial K- User Interference Channel

In the following section, a cyclic polynomial K-user interference channel is considered for $K \geq 3$ user-pairs. The K-user interference channel is of notable interest, since IA is capable to provide a number of $\frac{K}{2}$ DoF [3]. In contrast to the conventional approaches such a channel is not user-limited, since the DoF linearly scale with the number of user-pairs K.

3.4.1 Perfect Cyclic Interference Alignment

Similar to the initial exposition of the 2-user X- channel in Section 3.3, we use a symmetric messaging matrix $\boldsymbol{M} = \boldsymbol{I}_{K \times K}$. There is a number of $M = K$ independent messages $W_i = \boldsymbol{w}_{ii}$ dedicated to be conveyed from transmitter Tx_i to receiver Rx_i in disjoint pairs with indices $i \in \mathcal{K}$ for $\mathcal{K}_{\mathrm{Tx}} = \mathcal{K}_{\mathrm{Rx}} = \mathcal{K} = \{1, \ldots, K\}$. Note that in [3, Appendix I], the closely related example for IA by propagation delay yields a special case of this problem. We will not cover the general case with arbitrary message lengths in this dissertation. The K-user interference channel is depicted in Figure 3.9.

The channel matrix between is defined as $\boldsymbol{D} = (d_{ji})_{1 \leq i,j \leq K}$ with $d_{ji} \in \mathcal{D}$. This channel is also called *fully-connected* as each transmitter has a non-zero subchannel to each receiver. The polynomial $v_i(x)$ only contains the single (sub-)message W_i intended for Rx_i. The offset parameter $p_i \in \mathbb{N}_0$, for $i \in \mathcal{K}$, allocates the message W_i within $v_i(x)$:

$$v_i(x) = W_i x^{p_i}. \tag{3.99}$$

The input vector of the transmitted polynomials is denoted by $\boldsymbol{v} = (v_1(x), \ldots, v_K(x))$. The transfer function of channel yields the received vector $\boldsymbol{r} = (r_1(x), \ldots, r_K(x))$:

$$\boldsymbol{r}^{\mathsf{T}} = \boldsymbol{D}\boldsymbol{v}^{\mathsf{T}} \bmod (x^n - 1).$$

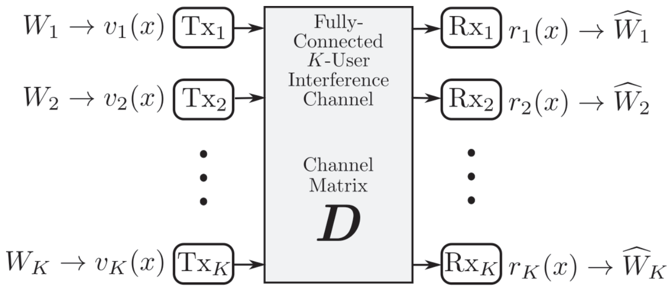


Figure 3.9: The fully-connected cyclic polynomial interference channel with K user-pairs and $M = K$ submessages $W_1, ..., W_K$, so that there is one message between each pair of transmitters $\mathrm{Tx}_1, ..., \mathrm{Tx}_K$ and receivers $\mathrm{Rx}_1, ..., \mathrm{Rx}_K$ for the transfer matrix $\boldsymbol{D}$.

And the received polynomial at Rx_j yields:

$$r_j(x) = \sum_{i=1}^{K} d_{ji} W_i x^{p_i} \bmod (x^n - 1).$$

The task of cyclic IA for the given K- user interference channel is to convey and decode the $M = K$ dedicated messages $W_1, ..., W_K$ interference-free within n dimensions. The scheme is optimal in the sense of cyclic IA if the number of dimensions n is minimal and still feasible.

In contrast to the 2×2 cyclic polynomial X- channel, neither the *multiple-access interference* nor the *intra-user interference conditions* are to be considered for this K- user interference channel, since there is only one message W_i per dedicated user-pair i anyway. Thus, only the following *inter-user interference conditions* are to be considered for $j \neq i \in \mathcal{K}$:

$$d_{jj} x^{p_j} \not\equiv d_{ji} x^{p_i} \bmod (x^n - 1). \tag{3.100}$$

We count $K(K-1)$ of these separability conditions in total for K users, i. e., a single receiver observes $K-1$ inter-user interference signals from the undesired transmitters. In the given case of K user-pairs, *perfect IA* means to align all $K-1$ interfering signals received at each Rx_j into a single dimension:

$$d_{ji} x^{p_i} \equiv d_{jk} x^{p_k} \bmod (x^n - 1), \tag{3.101}$$

with pairwise distinct $i, j, k \in \mathcal{K}$.

For notational convenience, we define auxiliary 2×2 submatrices of $\boldsymbol{D}$ denoted as $\boldsymbol{D}_{j,k,i}$, if the following structure is satisfied for pairwise distinct indices $i, j, k \in \mathcal{K}$:

$$\boldsymbol{D}_{j,k,i} = \begin{pmatrix} d_{jj} & d_{ji} \\ d_{kj} & d_{ki} \end{pmatrix}. \tag{3.102}$$

Theorem 3.9. *A perfect cyclic IA scheme for the K- user interference channel with messaging matrix $\boldsymbol{M} = \boldsymbol{I}_{K \times K}$ exists, if the three conditions:*

$\mathsf{det}(\boldsymbol{D}_{j,k,i}) \not\equiv 0 \bmod (x^n - 1)$,

$d_{ji}d_{kj}d_{ik} \equiv d_{ij}d_{jk}d_{ki} \bmod (x^n - 1)$,

and $n = 2$ *dimensions,*

hold for pairwise distinct $i, j, k \in \mathcal{K}$. *Then, cyclic IA achieves* $\frac{K}{2}$ *DoF.*

Proof:

(a) Necessity of $d_{ji}d_{kj}d_{ik} \equiv d_{ij}d_{jk}d_{ki} \bmod (x^n - 1)$, $n \in \mathbb{N}$:
Let $i, j, k \in \mathcal{K}$ be pairwise distinct. By relabelling the indices in (3.101) to $i \to j$, $j \to k$ and $k \to i$, we obtain (3.103). And by relabelling the indices in (3.101) to $i \to k$, $j \to i$ and $k \to j$, we obtain (3.104), respectively:

$$\begin{aligned} (3.101):\ d_{ji}x^{p_i} &\equiv d_{jk}x^{p_k} \bmod (x^n - 1), \\ d_{kj}x^{p_j} &\equiv d_{ki}x^{p_i} \bmod (x^n - 1), && (3.103) \\ d_{ik}x^{p_k} &\equiv d_{ij}x^{p_j} \bmod (x^n - 1). && (3.104) \end{aligned}$$

These three equivalences are solvable, if and only if the following condition holds:

$$d_{ji}d_{kj}d_{ik} \equiv d_{jk}d_{ki}d_{ij} \bmod (x^n - 1). \tag{3.105}$$

Otherwise, perfect IA cannot be applied.

(b) Necessity of $\mathsf{det}(\boldsymbol{D}_{j,k,i}) \not\equiv 0 \bmod (x^n - 1)$, $n \in \mathbb{N}$:
Inserting the relabelled (3.103) into the inter-user interference (3.100) condition yields:

$$\begin{aligned} d_{jj}d_{ki}d_{kj}^{-1}x^{p_i} &\not\equiv d_{ji}x^{p_i} \bmod (x^n - 1) \\ \Rightarrow d_{jj}d_{ki} &\not\equiv d_{kj}d_{ji} \bmod (x^n - 1) \\ \Rightarrow 0 &\not\equiv \mathsf{det}(\boldsymbol{D}_{j,k,i}) \bmod (x^n - 1), && (3.106) \end{aligned}$$

for pairwise distinct $i, j, k \in \mathcal{K}$. If the contraposition $\mathsf{det}(\boldsymbol{D}_{j,k,i}) \equiv 0 \bmod (x^n - 1)$ is assumed, the separability conditions can not be fulfilled by perfect IA.

(c) Necessity of $n > 1$ dimensions:
Only $n = 1$ dimension would preclude any separation of desired and interfering messages necessary for the inter-user interference conditions in (3.100).

(d) Sufficiency the given separability conditions with $n = 2$ *to achieve* $\frac{K}{2}$ *DoF*:
Firstly, we consider the valid channel matrix $\boldsymbol{D} = x^1\boldsymbol{I}_{K\times K} + (\boldsymbol{1}_{K\times K} - \boldsymbol{I}_{K\times K})x^2$ as in [3, Appendix I]. The condition (3.105) holds since any non-diagonal entry is x^2, i. e., $d_{ji}d_{kj}d_{ik} \equiv d_{ij}d_{jk}d_{ki} \equiv x^2x^2x^2 \equiv 1 \bmod (x^2 - 1)$. The condition (3.106) also holds since $\mathsf{det}(\boldsymbol{D}_{j,k,i}) \equiv x^3(x^1 - 1) \not\equiv 0 \bmod (x^2 - 1)$.

There are many other valid channel matrices $\boldsymbol{D}$: If all entries of a row in the given $\boldsymbol{D}$ are cyclically shifted by x^m, $m \in \mathbb{N}$, the two conditions still hold. For a shifted row j, we obtain $(d_{ji}x^m)d_{kj} \equiv d_{ki}(d_{jj}x^m) \bmod (x^2 - 1) \Rightarrow d_{ji}d_{kj} \equiv d_{ki}d_{jj} \bmod (x^2 - 1)$, leading to (3.106). An analogous argument holds for (3.105).

For such a valid $\boldsymbol{D}$, we fix p_1 w.l.o.g. and determine all other p_i, with $i = 2, ..., K$, by applying the perfect IA condition in (3.101):

$$x^{p_i} \equiv d_{ji}^{-1} d_{j1} x^{p_1} \bmod (x^2 - 1).$$

A solution that perfectly aligns all $K - 1$ interference signals at each receiver exists for the fixed offset parameters, since (3.105) holds by prerequisite on the given $\boldsymbol{D}$ for $n = 2$. The (aligned) interference signals do not overlap with the dedicated signals at all receivers, since the same perfect IA used for the derivation of the required (3.106) is also used here.

Altogether, K messages are conveyed interference-free in $n = 2$ dimensions and $\frac{K}{2}$ DoF are achieved. This corresponds to the upper bound (3.38) for messaging matrix $\boldsymbol{M} = \boldsymbol{I}_{K \times K}$. ■

The conditions on the K-user interference channel can also be expressed alternatively. We define yet another kind of auxiliary 2×2 - matrices of $\boldsymbol{D}$ with the following structure:

$$\boldsymbol{D}_{i,j} = \begin{pmatrix} d_{ii} & d_{ij} \\ d_{ji} & d_{jj} \end{pmatrix}, \tag{3.107}$$

for $i \neq j \in \mathcal{K}$. In contrast to the submatrices in (3.102), there are two diagonal elements of $\boldsymbol{D}$ in $\boldsymbol{D}_{i,j}$ instead of only one diagonal element in $\boldsymbol{D}_{j,k,i}$.

Corollary 3.10. *For perfect cyclic IA with $n = 2$, the following alternative conditions:*

$$\mathsf{det}(\boldsymbol{D}_{i,j}) \equiv 0 \bmod (x^2 - 1), \tag{3.108}$$

must hold for $i \neq j \in \mathcal{K}$.

Proof:
Since $n = 2$, a perfect IA scheme can be alternatively formulated by allocating dedicated and (aligned) interference signals side-by-side with:

$$d_{ii} x^{p_i} \equiv d_{ij} x^{p_j} x^1 \bmod (x^2 - 1), \tag{3.109}$$
$$d_{ii} x^{p_i} \equiv d_{ik} x^{p_k} x^1 \bmod (x^2 - 1), \tag{3.110}$$

for pairwise distinct $i, j, k \in \mathcal{K}$. On the one hand, we may substitute $d_{ii} x^{p_i}$ in both congruences and obtain (3.101) to derive condition (3.105) as above. On the other hand, we may consider a relabelled version of (3.109) with swapped indices instead:

$$\begin{aligned} (3.109): \ d_{ii} x^{p_i} &\equiv d_{ij} x^{p_j} x^1 \bmod (x^2 - 1), \\ d_{jj} x^{p_j} &\equiv d_{ji} x^{p_i} x^1 \bmod (x^2 - 1). \end{aligned} \tag{3.111}$$

Substituting these two congruences leads to the condition:

$$d_{ii} d_{jj} \equiv d_{ij} d_{ji} \bmod (x^2 - 1) \tag{3.112}$$
$$\Leftrightarrow \mathsf{det}(\boldsymbol{D}_{i,j}) \equiv 0 \bmod (x^2 - 1), \tag{3.113}$$

for $i \neq j \in \mathcal{K}$. ■

3.4.2 Common Eigenvectors in Separate Subspaces

It is shown in [3, Sec. IV-D], that perfect IA[6] is infeasible for the constant MIMO K-user interference channel with single antennas per user. The infeasibility of perfect IA is due to common eigenvectors in the dedicated and the interference subspaces. Interestingly, an analogous problem is recognizable in the CPCM, but it merely imposes a constraint on the $\boldsymbol{D}$. To briefly elaborate this, we consider a perfect cyclic IA scheme for the 3-user interference channel mimicking the MIMO IA scheme in [3, Section IV-D: Equations (25)-(27)]:

$$d_{13}x^{p_3} \equiv d_{12}x^{p_2} \bmod(x^2 - 1), \tag{3.114}$$
$$d_{13}x^{p_3} \equiv d_{31}x^{p_1} \bmod(x^2 - 1), \tag{3.115}$$
$$d_{32}x^{p_2} \equiv d_{31}x^{p_1} \bmod(x^2 - 1). \tag{3.116}$$

By resolving these equations w.r.t. x^{p_1}, we obtain:

$$x^{p_1} \equiv d_{13}^{-1}d_{23}d_{21}^{-1}d_{12}d_{32}^{-1}d_{31}x^{p_1} \bmod(x^2 - 1). \tag{3.117}$$

An analogous formulation arises in [3, Section IV-D], but with corresponding diagonal MIMO channel matrices and beam-forming vectors. This corresponding equation leads to the problem of common eigenvectors when applying perfect MIMO IA. In the cyclic IA scheme, (3.117) only implies a constraint like (3.105).

3.4.3 On the Feasibility Conditions of the K-User Interference Channel

To elaborate further common properties of the CPCM and the GMCM, we consider the feasibility conditions of the K-user MIMO interference channel with M_{Tx} transmit antennas and M_{Rx} receive antennas (cf. [39]). In the GMCM, let the channel matrices $\boldsymbol{H}_{ji}$ for each link from Tx_i to Rx_j have $M_{\mathrm{Rx}} \times M_{\mathrm{Tx}}$ dimensions. The transmit beam-forming matrices $\boldsymbol{V}_i$ have $M_{\mathrm{Tx}} \times d_i$ dimensions and the receive filtering matrices $\boldsymbol{N}_i$ have $M_{\mathrm{Rx}} \times d_i$ dimensions. We have $\mathsf{rank}(\boldsymbol{V}_i) = d_i = \mathsf{rank}(\boldsymbol{N}_i) = d_i \leq \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})$ for both matrices. The feasibility conditions for this case are compactly formulated by:

$$\boldsymbol{N}_j\boldsymbol{H}_{ji}\boldsymbol{V}_i = \mathbf{0}_{d_j \times d_i}, \text{ for all } i \neq j \in \mathcal{K}, \tag{3.118}$$
$$\mathsf{rank}(\boldsymbol{N}_i\boldsymbol{H}_{ii}\boldsymbol{V}_i) = d_i, \text{ for all } i \in \mathcal{K}, \tag{3.119}$$

for some properly chosen beam-forming and zero-forcing matrices, as in [39], [40]. The first condition (3.118) ensures that the interference signals from Tx_i are projected into the null space of the receiver at Rx_j for distinct i, j. The second condition (3.119) ensures that the dedicated signals from Tx_i are projected into a dedicated signal space of d_i dimensions at Rx_i. Analogous to the demanded properties in Section 2.2.3, satisfying these feasibility conditions also provides linear interference-free communication and linear decodability of all dedicated signals.

[6] However, *asymptotically perfect IA* is feasible using symbol-extensions for instance [3].

We now translate these feasibility conditions to an analogous expression for the CPCM. Let a *zero-forcing polynomial* at Rx_j be defined by:

$$z_j(x) = \sum_{k=0}^{n-1} z_j^{[k]} x^k, \;\; z_j^{[k]} \in \{0,1\}. \tag{3.120}$$

By element-wise multiplying the zero-forcing polynomial $z_j(x)$ with the received polynomial $r_j(x)$, the resulting polynomial $z_j(x) \circ r_j(x)$ is filtered correspondingly. A filter coefficient $z_j^{[k]} = 1$ keeps the received coefficient $r_j^{[k]}$ at offset k unchanged, and a filter coefficient $z_j^{[k]} = 0$ zero-forces the received coefficient at offset k.

With such zero-forcing polynomials, the feasibility conditions for the cyclic polynomial K-user interference channel are analogously formalized as follows:

$$z_j(x) \circ d_{ji} v_i(x) = 0, \;\text{for } i \neq j \in \mathcal{K}, \tag{3.121}$$

$$\mathsf{rank}(z_i(x) \circ d_{ii} v_i(x)) = m_{ii}. \tag{3.122}$$

The first condition provides that interference space is zero-forced at each receiver, and the second condition provides that there are sufficiently dimensions available for the m_{ii} dedicated submessages from Tx_i to Rx_i, for all $i \in \mathcal{K}$. These conditions are fulfilled by the cyclic IA scheme proven in Theorem 3.9 using the zero-forcing polynomial:

$$z_j(x) \equiv d_{ji} x^{p_i} \equiv d_{jk} x^{p_k} \mod(x^2 - 1). \tag{3.123}$$

3.5 Infeasibility of Perfect Cyclic IA in Multi-User X-Networks

In this section, we consider a 3-user X-network, an extension of the 2-user X-channel. Similar to the 2-user X-network, each transmitter has one dedicated message for each receiver, i. e., three messages per transmitter. The investigation of this channel particularly addresses an interesting open problem that was stated in the discussion of [16, Section V]:

Problem: *Is perfect IA by propagation delay feasible on K-user X-networks with $K > 2$ users?*

We will show in this section that, in terms of the CPCM, perfect cyclic IA in a K-user X - network for $K > 2$ users is indeed overconstrained and hence infeasible. We observe that this property even goes beyond the closely related problem of common invariant subspaces of constant MIMO IA [37].

Then, in order to tackle this infeasibility problem, we extend the capabilities of the 3-user X-network by using a cooperative *interference alignment and cancellation (IAC)* scheme over rate-limited *backhaul network (BHN)*. The concept of IAC is initially introduced in [61] and also applied in [62] for instance. Therein, a BHN provides a limited exchange of messages at the receiver-side to support the cancellation of known interference by the aid of cognitive messages from the other cooperating receivers.

We first analyze a simple feedforward scheme achieving $\frac{9}{5}$ DoF within 5 dimensions and only one message over an interference-free feedforward BHN. In other words, we permit a single cognitive receiver that has interference-free access to one message of the transmitters.

Our second step leads us to a related cyclic IAC scheme, but its cognitive messages are restricted to a receiver-sided backhaul network (Rx-BHN). In that case, only receivers can cooperate and exchange a limited number of messages. Cyclic IAC achieves the $\frac{9}{5}$ DoF with a minimum of only 2 messages over the Rx-BHN.

For a transmitter-sided backhaul network (Rx-BHN), we employ the concept of IN as discussed in [47] and [63] to inhibit interference at the receivers. IN is a communication scheme cancelling interference 'over the air' by aligning complementary versions of the same message within the same signalling space. Since IN is not in the main focus of this particular section, we point the reader to Sections 4.1 and 4.2 in the subsequent chapter. We observe a duality relationship between IAC on the Rx-BHN and IN on the dual Tx-BHN. This insight also generalizes the related observations in [23] for a 2IFC with cooperation.

3.5.1 Cyclic Polynomial 3-User X-Network

Note that, we already derived the DoF for the 2-user X-channel in Section 3.3, and in [27, Section III]. Here, each transmitter intends to convey $K = 3$ dedicated messages, one to each receiver. The set of user-indices is $\mathcal{K}_{\mathrm{Tx}} = \mathcal{K}_{\mathrm{Rx}} = \mathcal{K} = \{1, 2, 3\}$. For the sake of simplicity, we constrain our model to $\boldsymbol{M} = \mathbf{1}_{3\times 3}$, i. e., $m_{ji} = 1$ submessages for all transmitter-receiver pairs. We set our focus on perfect IA. There are $M = K^2 = 9$ unicast messages in the system in total for these given assumptions. The transmitted signal from Tx_i is a polynomial with (sub-)messages W_{ji} dedicated for receiver Rx_j:

$$v_i(x) = \textstyle\sum_{j\in\mathcal{K}} W_{ji} x^{p_{ji}}. \tag{3.124}$$

Each receiver must decode three dedicated messages, and receives six interfering signals, i.e., they must cope with two interfering messages per transmitter Tx_i. The received signal $r_j(x)$ at Rx_j is:

$$r_j(x) \equiv \textstyle\sum_{i\in\mathcal{K}} d_{ji} v_i(x) \bmod (x^n - 1). \tag{3.125}$$

The 3-user X-network is depicted in Figure 3.10.

Separability Conditions

In order to guarantee decodability, all three types of *separability conditions* as discussed in Section 3.3.1 must be considered for the 3-user X-network. Including all nine messages, the intra-user interference conditions at Tx_i, for pairwise distinct p_{ii}, p_{ji}, p_{ki}, are:

$$x^{p_{ji}} \not\equiv x^{p_{ki}} \bmod (x^n - 1), \tag{3.126}$$
$$x^{p_{ii}} \not\equiv x^{p_{ji}} \bmod (x^n - 1), \tag{3.127}$$
$$x^{p_{ii}} \not\equiv x^{p_{ki}} \bmod (x^n - 1). \tag{3.128}$$

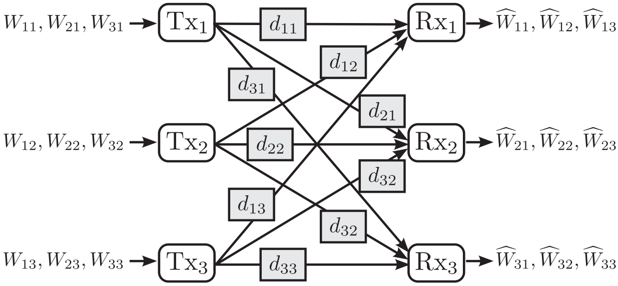


Figure 3.10: The fully-connected 3-user X-network with three transmitters Tx_1, Tx_2 and Tx_3, three receivers $\text{Rx}_1, \text{Rx}_2, \text{Rx}_3$, nine independent messages W_{ji} and nine estimated messages $\widehat{W}_{ji}$. The influence of the channel is parameterized by d_{ji}.

The multiple-access interference conditions at Rx_i for distinct dedicated signals $d_{ii}x^{p_{ii}}$, $d_{ij}x^{p_{ij}}$, $d_{ik}x^{p_{ik}}$, are:

$$d_{ij}x^{p_{ij}} \not\equiv d_{ik}x^{p_{ik}} \mod (x^n - 1), \tag{3.129}$$
$$d_{ii}x^{p_{ii}} \not\equiv d_{ij}x^{p_{ij}} \mod (x^n - 1), \tag{3.130}$$
$$d_{ii}x^{p_{ii}} \not\equiv d_{ik}x^{p_{ik}} \mod (x^n - 1). \tag{3.131}$$

And the inter-user interference conditions at Rx_i are:

$$d_{ii}x^{p_{ii}} \not\equiv d_{ij}x^{p_{kj}} \mod (x^n - 1), \tag{3.132}$$
$$d_{ii}x^{p_{ii}} \not\equiv d_{ij}x^{p_{jj}} \mod (x^n - 1), \tag{3.133}$$
$$d_{ii}x^{p_{ii}} \not\equiv d_{ik}x^{p_{jk}} \mod (x^n - 1), \tag{3.134}$$
$$d_{ii}x^{p_{ii}} \not\equiv d_{ik}x^{p_{kk}} \mod (x^n - 1), \tag{3.135}$$
$$d_{ij}x^{p_{ij}} \not\equiv d_{ii}x^{p_{ji}} \mod (x^n - 1), \tag{3.136}$$
$$d_{ij}x^{p_{ij}} \not\equiv d_{ii}x^{p_{ki}} \mod (x^n - 1), \tag{3.137}$$
$$d_{ij}x^{p_{ij}} \not\equiv d_{ik}x^{p_{jk}} \mod (x^n - 1), \tag{3.138}$$
$$d_{ij}x^{p_{ij}} \not\equiv d_{ik}x^{p_{kk}} \mod (x^n - 1), \tag{3.139}$$

for distinct indices $i, j, k \in \mathcal{K}$, respectively. Note that by a circular relabelling of indices, these conditions can also be expressed for Tx_j, Tx_k, Rx_j and Rx_k (cf. Appendix A).

3.5.2 Infeasibility Problem of Perfect Cyclic Interference Alignment

A receiver Rx_i will receive a superposition of six interfering signals in total: $d_{ii}x^{p_{ji}}$, $d_{ii}x^{p_{ki}}$, $d_{ij}x^{p_{kj}}$, $d_{ij}x^{p_{jj}}$, $d_{ik}x^{p_{jk}}$, $d_{ik}x^{p_{kk}}$. Two interfering signals from the same transmitter can not be aligned due to the intra-user interference conditions (3.126) to (3.128). As three dimensions are reserved for dedicated signals and at least two dimensions must be reserved for interference, we demand $n \geq 5$ dimensions.

Perfect cyclic IA is optimal and requires $n = 5$ dimensions. Three interference signals, i.e., one from each transmitter, must be aligned to a single dimension reserved for interference only. A potentially perfect cyclic IA scheme at Rx_i is constructed by choosing one element from each of these three sets as implied by curly brackets:

$$\left\{ \begin{array}{c} d_{ii}x^{p_{ji}} \\ d_{ii}x^{p_{ki}} \end{array} \right\} \equiv \left\{ \begin{array}{c} d_{ij}x^{p_{jj}} \\ d_{ij}x^{p_{kj}} \end{array} \right\} \equiv \left\{ \begin{array}{c} d_{ik}x^{p_{jk}} \\ d_{ik}x^{p_{kk}} \end{array} \right\} \mod (x^5 - 1), \tag{3.140}$$

e. g., by taking the elements in the first row, we obtain:

$$d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{jk}} \mod (x^5 - 1), \tag{3.141}$$

for a fully symmetric perfect IA scheme similar to the one in [37, Section V-C]. IA in one interference dimension directly implicates the complementary alignment of the other interference dimension at the same receiver. For the example given in (3.141), the complementary alignment is given by the second row of (3.140):

$$d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}} \equiv d_{ik}x^{p_{kk}} \mod (x^5 - 1).$$

For notational convenience, we will denote auxiliary submatrices of $\boldsymbol{D}$ by:

$$\boldsymbol{D}_{i,k,j,l} = \begin{pmatrix} d_{ij} & d_{il} \\ d_{kj} & d_{kl} \end{pmatrix}.$$

Note that the determinant of $\boldsymbol{D}_{i,k,j,l}$ also implies the following symmetries:

$$\mathsf{det}(\boldsymbol{D}_{i,k,j,l}) \equiv \mathsf{det}(\boldsymbol{D}_{k,i,l,j}) \equiv -\mathsf{det}(\boldsymbol{D}_{k,i,j,l}) \equiv -\mathsf{det}(\boldsymbol{D}_{i,k,l,j}) \mod (x^n - 1).$$

Theorem 3.11. *Perfect cyclic IA with the messaging matrix $\boldsymbol{M} = \mathbf{1}_{3\times 3}$ is infeasible on the 3-user X-network.*

Proof:
Each $\boldsymbol{D}_{i,k,j,l}$ corresponds to a subordinate 2×2 X-channel matrix with distinct transmitters Tx_j, Tx_l, and distinct receivers Rx_i, Rx_k. Note that for the 2×2 X-channel, a non-zero determinant of the channel matrix is necessary to perform IA, as we have already shown in [27, Theorem 1 (a)].

Now, we assume that, e. g., $\mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \equiv 0 \mod (x^5 - 1)$ holds and consider some particular implications on (3.140). On the one hand, if $d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}} \mod (x^5 - 1)$ holds, the above assumption leads to $d_{jj}x^{p_{jj}} \equiv d_{ji}x^{p_{ji}} \mod (x^5 - 1)$. But this contradicts the multiple-access interference conditions. In analogy, aligning $d_{jj}x^{p_{ij}} \equiv d_{ji}x^{p_{ii}} \mod (x^5 - 1)$ would imply $d_{ii}x^{p_{ii}} \equiv d_{ij}x^{p_{ij}} \mod (x^5 - 1)$, yielding another violation of the separability conditions.

Contrariwise, by aligning $d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}}$ with the initial assumption $\mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \equiv 0$ would implies $d_{ji}x^{p_{ki}} \equiv d_{jj}x^{p_{kj}}$. This is not a contradiction so far. Beyond that, the above assumption is even a necessary condition when both of these two alignments are used for the IA scheme.

However, the complementary alignment at Rx_i to $d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}} \mod (x^5 - 1)$ is again the previously considered alignment $d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}} \mod (x^5 - 1)$. Thus, the separability conditions are violated for both $\mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \equiv 0$ and $\mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \not\equiv 0$. In other words, there is no feasible $\boldsymbol{D}$ available. This conflict carries over to all minors of $\boldsymbol{D}$ analogously. ∎

Theorem 3.11 entails that perfect cyclic IA is also infeasible for the K-user X-network with $K \geq 3$ users since there are $\binom{K}{3}$ embedded 3-user X-networks.

Nonetheless, this problem does not exclude imperfect cyclic IA schemes with a number of $n > 2K - 1$ dimensions. But imperfect cyclic IA schemes do not achieve the upper bound (3.38) exactly. A corresponding approach will not be discussed in this dissertation.

3.5.3 Common Eigenvectors in Separate Subspaces - Revisited

The infeasibility of perfect cyclic IA shown here is also linked to the problem of the common eigenvectors in perfect spatial IA schemes as discussed in Section 3.4.2 and in the works [3, Section IV-D], [64], and [37, Section V-C].

But the common eigenvector problem is only a subordinate part of the infeasibility problem presented in Theorem 3.11 above. To briefly elaborate this, we consider a symmetric perfect cyclic IA scheme which is analogous to the spatial IA scheme in [37, Equations (10)-(12)]:

$$d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{jk}} \bmod (x^5 - 1), \tag{3.142}$$
$$d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{kj}} \equiv d_{ik}x^{p_{kk}} \bmod (x^5 - 1), \tag{3.143}$$

for pairwise distinct indices $i, j, k \in \mathcal{K}$. Due to symmetry, the alignment of (3.142) at receiver Rx_k corresponds to:

$$d_{kk}x^{p_{jk}} \equiv d_{kj}x^{p_{jj}} \equiv d_{ki}x^{p_{ji}} \bmod (x^5 - 1), \tag{3.144}$$

by a simple relabelling of indices. With (3.142), (3.144), we obtain:

$$\begin{aligned} d_{ii}x^{p_{ji}} &\equiv d_{ij}x^{p_{jj}} \bmod (x^5 - 1), \\ d_{ki}x^{p_{ji}} &\equiv d_{kj}x^{p_{jj}} \bmod (x^5 - 1), \\ \Rightarrow d_{ii}d_{ij}^{-1}d_{kj}d_{ki}^{-1}x^{p_{ji}} &\equiv x^{p_{ji}} \bmod (x^5 - 1). \end{aligned} \tag{3.145}$$

An analogous formulation arises in [37, Equation (15)] with corresponding diagonal MIMO channel matrices and leads to the problem of common eigenvectors in perfect spatial IA. But in the case of cyclic IA, the result of (3.145) only implies the additional constraint $\mathsf{det}(\boldsymbol{D}_{i,k,j,i}) \equiv 0 \bmod (x^5 - 1)$.

3.6 3-User X-Networks with Minimal Backhaul

Since perfect cyclic IA is already shown to be an overconstrained problem, our upcoming approach is to relax the restraining conditions by providing a limited number of cognitive messages over a BHN to achieve sufficient feasibility.

3.6.1 Cyclic IA with Minimal Feedforward Backhaul Networks

A first and very intuitive approach is to include a *feedforward (FF) backhaul network* between some transmitters and receivers. A single FF-link $\theta_{\mathrm{FF},ji}$ between Tx_i and Rx_j with rate $\Theta_{\mathrm{FF},ji} = 1$ simply bypasses the channel d_{ji} for a single message so that the actual transmission of the forwarded message may be omitted. The FF-BHN is also shown at the bottom of Figure 5.4. Note that this approach corresponds to using one cognitive receiver Rx_j knowing W_{ji}.

We propose the following alignment scheme and prove its optimality w. r. t. the minimal necessary sum-rate Θ_{FF}:

At Rx_i, interference from Rx_j and Rx_k is perfectly aligned within two dimensions:

$$d_{ii}x^{p_{ji}} \equiv d_{ij}x^{p_{kj}} \equiv d_{ik}x^{p_{jk}} \bmod (x^5-1), \tag{3.146}$$
$$d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{kk}} \bmod (x^5-1). \tag{3.147}$$

The dedicated and interfering signals at Rx_j are aligned by:

$$d_{ji}x^{p_{ki}} \equiv d_{jj}x^{p_{kj}} \equiv d_{jk}x^{p_{ik}} \bmod (x^5-1), \tag{3.148}$$
$$d_{ji}x^{p_{ii}} \equiv d_{jk}x^{p_{kk}} \bmod (x^5-1), \tag{3.149}$$
$$d_{jj}x^{p_{ij}} \equiv d_{jk}x^{p_{jk}} \bmod (x^5-1), \tag{3.150}$$

and similarly, we use the following Cyclic IA scheme at Rx_k:

$$d_{ki}x^{p_{ii}} \equiv d_{kj}x^{p_{jj}} \equiv d_{kk}x^{p_{ik}} \bmod (x^5-1), \tag{3.151}$$
$$d_{kj}x^{p_{ij}} \equiv d_{ki}x^{p_{ji}} \bmod (x^5-1), \tag{3.152}$$
$$d_{ki}x^{p_{ki}} \equiv d_{kk}x^{p_{jk}} \bmod (x^5-1). \tag{3.153}$$

A relabelling of indices is not permitted in this asymmetric IA scheme. Note that (3.150) and (3.153) explicitly violate the separability conditions. Independent of channel matrix $\boldsymbol{D}$, the dedicated messages W_{jk} and W_{ki} can not be decoded yet.

Theorem 3.12. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by Cyclic IA with feedforward for $\min(\Theta_{\mathrm{FF}}) \geq 1$.*

For the considered Cyclic IA scheme, we assume that the following conditions:

(i) $d_{ij}d_{ki}d_{jk} \equiv d_{ji}d_{ik}d_{kj} \bmod (x^5-1)$,

(ii) $d_{ii}d_{jk}d_{kj} \equiv d_{jj}d_{ik}d_{ki} \equiv d_{kk}d_{ij}d_{ji} \bmod (x^5-1)$,

(iii) $\det(\boldsymbol{D}_{i,k,i,k}) \not\equiv 0 \bmod (x^5-1)$,

(iv) $\det(\boldsymbol{D}_{i,j,i,j}) \not\equiv 0 \bmod (x^5-1)$,

(v) $\det(\boldsymbol{D}_{i,j,i,k}) \not\equiv 0 \bmod (x^5-1)$,

(vi) $\det(\boldsymbol{D}_{i,j,j,k}) \not\equiv 0 \bmod (x^5-1)$,

(vii) $\det(\boldsymbol{D}_{k,j,k,i}) \not\equiv 0 \bmod (x^5-1)$,

(viii) $\det(\boldsymbol{D}_{k,j,k,j}) \not\equiv 0 \bmod (x^5-1)$,

(ix) $d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{jk}d_{ki} \equiv d_{ji}d_{kj}d_{ik} \mod (x^5 - 1)$,

(x) $d_{kk}d_{ii}d_{jj}d_{kk} \not\equiv d_{jk}d_{kj}d_{ik}d_{ki} \mod (x^5 - 1)$,
$d_{ii}d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{ji}d_{ik}d_{ki} \mod (x^5 - 1)$,
$d_{jj}d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{ji}d_{jk}d_{kj} \mod (x^5 - 1)$,

hold for distinct indices $i, j, k \in \mathcal{K}$.

Proof:
(a) *Necessity of* $\Theta_{FF} \geq 1$:
Since $\Theta_{\mathrm{FF}} = 0$ would correspond to using no FF-link at all, the necessity of $\Theta_{\mathrm{FF}} \geq 1$ follows from Theorem 3.11 evidently.

(b) *Necessity of constraints (i) to (x) for the given scheme*:
First, we consider the constraints (i), (ii) that are implied by the cyclic IA scheme given in (3.146) to (3.153). We depict how the parameters are interlinked by the adjacency graph shown in Figure A.1 of Appendix A. Other potentially valid alignment schemes will imply a different set of constraints and hence another adjacency graph.

Constraint (i) is obtained by substituting the parameters p_{jj}, p_{kk}, p_{ii} in (3.147), (3.148) and (3.151).

One part of constraint (ii) yields from substituting p_{jk}, p_{ji}, p_{ij} in (3.146), (3.150), (3.149). Another part from (ii) yields from substituting p_{jj}, p_{kj}, p_{ik} in (3.147), (3.151), (3.148), and the remaining part from substituting p_{kj}, p_{ki}, p_{jk} in (3.148), (3.153), (3.146).

Now, we consider the impact of the separability conditions. Note that the feasibility of (3.150) and (3.153) is provided by the FF-BHN, as we will show in part (b), i. e., these particular violations are excluded. Constraint (iii) is derived from (3.131) and (3.151). Constraint (iv) is derived by substituting p_{ki}, p_{ji}, p_{kj} with (3.126), (3.146), (3.148).

The remaining constraints (v) to (x) are proven analogously and w. r. t. all separability conditions at each receiver. The extensive proof of those constraints is given in Appendix A. Note that the three constraints of (x) are equivalent due to (ii).

(b) *Sufficiency of Cyclic IA with* $\Theta_{\mathrm{FF}} = 1$:
It suffices to bypass the transmission of the dedicated message W_{jk} over d_{ji} through a single FF-link $\theta_{\mathrm{FF},jk}$ with sum-rate $\Theta_{FF} = 1$. Then, Tx_k may omit the transmission of W_{jk} over $\boldsymbol{D}$, and Rx_j can still decode W_{jk} from the FF-BHN. As W_{jk} is not transmitted over $\boldsymbol{D}$ at all, Rx_k can also decode W_{ki} interference-free.

To show that the proposed cyclic IA scheme with FF is feasible now, all nine transmission parameters must be resolved. As indicated by the adjacency graph in Figure A.1, we fix the topmost parameter p_{ki}, w.l.o.g. With (3.147), we obtain p_{jj}, p_{kk}, (3.148) provides p_{kj}, p_{ik}, and (3.153) yields p_{jk}. With (3.149), p_{ii} yields from p_{kk}. With (3.146), p_{ji} yields from p_{kj}. And with (3.150), p_{ij} yields from p_{jk}.

A valid matrix, normalized w. r. t. the main diagonal is, e. g.:

$$\boldsymbol{D} = \begin{pmatrix} 1 & x^4 & x^2 \\ x^4 & 1 & x^2 \\ x & x & 1 \end{pmatrix},$$

as all 10 constraints (i) $\equiv x^2$, (ii) $\equiv x^3$, (iii) $\equiv 1 - x^3$, (iv) $\equiv 1 - x^3$, (v) $\equiv x^2 - 1$, (vi) $\equiv x^1 - x^2$, (vii) $\equiv x^4 - x^3$, (viii) $\equiv 1 - x^3$, (ix) $1 \not\equiv x^2$, (x) $1 \not\equiv x^1$ for $i = 1$, $j = 2$, $k = 3$, are fulfilled. A valid set of transmission parameters satisfying all conditions with a fixed $p_{31} = 4$ is:

$$(p_{11}, p_{21}, p_{31}, p_{12}, p_{22}, p_{32}, p_{13}, p_{23}, p_{33}) = (0, 2, 4, 2, 0, 3, 1, \varnothing, 2).$$

The transmitted signals are:

$$\begin{aligned} v_1(x) &= W_{11}x^0 + W_{21}x^2 + W_{31}x^4, \\ v_2(x) &= W_{12}x^2 + W_{22}x^0 + W_{32}x^3, \\ v_3(x) &= W_{13}x^1 + W_{33}x^2, \end{aligned}$$

and the received signals are:

$$\begin{aligned} r_1(x) &\equiv W_{11}x^0 + W_{12}x^1 + (W_{21} + W_{32})x^1 + W_{13}x^3 + (W_{31} + W_{22} + W_{13})x^4 \mod (x^n - 1), \\ r_2(x) &\equiv W_{22}x^0 + W_{21}x^1 + W_{12}x^2 + (W_{13} + W_{32} + W_{31})x^3 + (W_{11} + W_{33})x^4 \mod (x^n - 1), \\ r_3(x) &\equiv W_{31}x^0 + (W_{11} + W_{22} + W_{13})x^1 + W_{33}x^2 + (W_{12} + W_{21})x^3 + W_{32}x^4 \mod (x^n - 1). \end{aligned}$$

Altogether, including the forwarded message W_{23} for Rx_2 over the FF-BHN, $\frac{9}{5}$ DoF are achieved by cyclic IA with $\Theta_{\mathrm{FF}} = 1$. ■

Note that a delayed FF transmission only delays the decoding time, but it does not affect the feasibility.

We would like to emphasize, that the given constraints on $\boldsymbol{D}$ are profoundly interdependent with the proposed IA scheme. Nonetheless, the analysis of comparable Cyclic IA schemes on this channel can be performed analogously to Theorem 3.12.

3.6.2 Cyclic IAC over Minimal Receiver Backhaul Networks

Now, instead of a FF-BHN, we consider a *receiver backhaul network* (Rx-BHN). The Rx-BHN only permits that receivers may exchange messages to resolve leaking interference in order to satisfy all separability conditions. The Rx-BHN is depicted in Figure 5.4 on the right-hand side. A single link with rate $\Theta_{\mathrm{Rx},ij}$ in the Rx-BHN from Rx_i to Rx_j is denoted by $\theta_{\mathrm{Rx},ji}$.

Similar to the previous section, our aim is to characterize the minimal sum-rate Θ_{Rx} on the Rx-BHN, that is necessary to achieve the upper bound of $\frac{9}{5}$ DoF on the 3-user X-network.

Lemma 3.13. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by cyclic IAC with $\Theta_{\mathrm{Rx}} \geq 2$ for the conditions given in Theorem 3.12.*

Proof:

(a) *Necessity of* $\Theta_{\mathrm{Rx}} \geq 2$*:*
As cyclic IA without cancellation is precluded by Theorem 3.11 for $n = 5$, it follows that $\Theta_{\mathrm{Rx}} > 0$. In contrast to Theorem 3.12, no message can be neglected and bypassed so that all must be sent over the channel $\boldsymbol{D}$. Thus the case $\Theta_{\mathrm{Rx}} = 1$ demands that

the interference at two receivers, say, Tx_i and Tx_j, must be perfectly aligned and only one interfering signal may leak at the remaining receiver Tx_k. However, simultaneous perfect cyclic IA at two receivers is already precluded by Theorem 3.11.

(b) *Sufficiency of Cyclic IAC for* $\Theta_{\mathrm{Rx}} = 2$*:*
We consider the same cyclic IA scheme as provided in (3.146) to (3.153) subject to constraints (i) to (x) of Theorem 3.12. Note that message W_{jk} is indeed transmitted here. But now, the leaking interference in (3.150) and (3.153) is resolved by $\Theta_{\mathrm{Rx}} = 2$ messages over the Rx-BHN. In particular W_{ij} is conveyed over $\theta_{\mathrm{Rx},ji}$, so that W_{ij} can be cancelled from the aligned $W_{ij} + W_{jk}$ to decode the dedicated message W_{jk} at Rx_j. In a subsequent step, W_{jk} is conveyed over $\theta_{\mathrm{Rx},kj}$, so that W_{jk} is cancelled from the aligned $W_{jk} + W_{ki}$ to decode the dedicated message W_{ki} at Rx_k. ■

A delayed Rx-BHN transmission does not affect feasibility as long as the backhaul messages W_{ij} and W_{jk} for cancellation adhere to the proposed sequence.

3.6.3 Cyclic IN over Minimal Transmitter Backhaul Networks

We now consider the reversed case: Transmitters are connected via a *transmitter backhaul network* (Tx-BHN) instead. A backhaul link from Tx_i to Tx_j is described by $\theta_{\mathrm{Tx},ji}$ correspondingly. The sum-rate over the Tx-BHN is denoted by Θ_{Tx}. The Tx-BHN is depicted in Figure 5.4 on the left-hand side.

Lemma 3.14. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3 - user X - network is achievable by cyclic IN with $\Theta_{\mathrm{Tx}} \geq 2$ for the conditions given in Theorem 3.12.*

Proof:
This scheme is a dual to Lemma 3.13 for the Rx-BHN as considered above, so that the necessity of $\Theta_{\mathrm{Tx}} \geq 2$ is analogous. Again, we use the alignment scheme of (3.146) to (3.153) subject to constraints (i) to (x) of Theorem 3.12. But in contrast to Lemma 3.13, W_{ij} is firstly conveyed over $\theta_{\mathrm{Tx},kj}$ with $\Theta_{\mathrm{Tx},kj} = 1$ and then the combined message $W_{ij} - W_{jk}$ is conveyed over $\theta_{\mathrm{Tx},ik}$ with $\Theta_{\mathrm{Tx},ik} = 1$. Tx_k transmits the superposition $W_{jk} - W_{ij}$ instead of W_{jk} only, and Tx_i transmits the superposition $W_{ij} - W_{ki} - W_{jk}$ instead of W_{ki} only. This change does not only maintain the decodability of the dedicated signals received at Rx_i or Rx_j, but rather neutralizes the previously leaking interference observed at Rx_j and Rx_k. ■

In contrast to cyclic IAC over the Rx-BHN, the exchange of signals over the Tx-BHN must be performed at any time before the actual transmission. This relationship also endorses a related IAC-IN duality property reported in [23] for a linear deterministic 2-user interference channel with cooperation.

3.6.4 Combined Cyclic IAC and IN over Minimal Tx/Rx-BHNs

Moreover, if transmitters and receivers are each connected to a disjoint Tx-BHN and Rx-BHN, the IAC and IN schemes of Sections 3.6.2 and 3.6.3 can be combined. The sum-rate over both BHNs is denoted by $\Theta_{\mathrm{TR}} = \Theta_{\mathrm{Rx}} + \Theta_{\mathrm{Tx}}$.

	x^0	x^1	x^2	x^3	x^4
$v_1(x)$	W_{11}	0	W_{21}	0	W_{31}
$v_2(x)$	W_{22}	0	W_{12}	W_{32}	0
$v_3(x)$	W_{23}	W_{13}	W_{33}	0	0
$r_1(x)$	W_{11}	W_{12}	$W_{21}+W_{32}+W_{23}$	W_{13}	$W_{31}+W_{22}+W_{33}$
$r_2(x)$	W_{22}	W_{21}	$W_{23}+W_{12}$	$W_{31}+W_{32}+W_{13}$	$W_{11}+W_{33}$
$r_3(x)$	$W_{31}+W_{23}$	$W_{11}+W_{22}+W_{13}$	W_{33}	$W_{21}+W_{12}$	W_{32}

Figure 3.11: Example of cyclic IAC: The transmitted signals are $v_i(x)$ and the received signals are $r_i(x)$, $i \in \mathcal{K}$. Dedicated signals are highlighted in blue and interference signals in red. Cyclic right-shifts are used here.

Corollary 3.15. *The upper bound of $\frac{9}{5}$ DoF for $n = 5$ on the 3-user X-network is achievable by cyclic IAC/IN with $\Theta_{\mathrm{TR}} \geq 2$.*

Proof:

Using (3.146) to (3.153), W_{ij} is provided over $\theta_{\mathrm{Tx},kj}$ and $W_{jk}-W_{ij}$ replaces W_{ij} at Tx_k so that W_{ij} is neutralized at Rx_j. Rx_k receives $W_{ki}+W_{jk}-W_{ij}$. Then, $W_{jk}+W_{ji}$ is provided over $\theta_{\mathrm{Rx},kj}$. Rx_k decodes $(W_{ki}+W_{jk}-W_{ij})-(W_{jk}+W_{ji})+(W_{ji}+W_{ij}) = W_{ki}$ using its interfered signal and (3.152). ∎

3.6.5 Examples for the Given Cyclic IA, IAC and IN Schemes

We use the following valid channel matrix for all examples in this section:

$$\boldsymbol{D} = \begin{pmatrix} 1 & x^4 & x^2 \\ x^4 & 1 & x^2 \\ x & x & 1 \end{pmatrix}, \tag{3.154}$$

and we fix the indices $i = 1, j = 2, k = 3$ and the parameter vector:

$$\boldsymbol{p} = (0,2,4,2,0,3,1,0,2),$$

as in part (b) of the proof for Theorem 3.12. The resulting transmitted and received signals for this example are depicted in the table of Figure 3.11. It is easy to see that all dedicated messages are decodable at $\mathrm{Rx}_1, \mathrm{Rx}_2$ and Rx_3, except the messages W_{23} at Rx_2 and W_{31} at Rx_3.

Cyclic IAC with a FF-BHN

According to Theorem 3.12, Tx_3 provides W_{23} to Rx_2 via the FF-BHN, so that W_{23} is decodable at Rx_2 now. As the message W_{23} is not transmitted over the channel $\boldsymbol{D}$ at all, W_{31} is also decodable at Rx_3.

Cyclic IAC with a Rx-BHN

According to Lemma 3.13, Rx_1 provides W_{12} to Rx_2 via the Rx-BHN, so that W_{22} is decodable at Rx_2 now. After decoding W_{23} at Rx_2, Rx_2 provides W_{23} to Rx_3 via the Rx-BHN, so that W_{31} is decodable at Rx_3.

Cyclic IN with a Tx-BHN

According to Lemma 3.14, Tx_2 provides W_{12} to Tx_3 via the Tx-BHN, and then Tx_3 provides the combined message $W_{12} - W_{23}$ to Tx_1 before the actual transmission of these messages over $\boldsymbol{D}$. Then, Tx_3 transmits $W_{23} - W_{12}$ instead of W_{23}, and Tx_1 transmits $W_{12} - W_{31} - W_{23}$ instead of W_{31}. As a result, all interfering messages neutralize each other and all dedicated messages are decodable.

Cyclic IAC and IN with Tx/Rx-BHNs

According to Corollary 3.15, Tx_2 provides W_{12} to Tx_3 over the Tx-BHN before transmission. Tx_3 transmits $W_{23} - W_{12}$ instead of W_{23}. On the receiver-side, Rx_2 can decode W_{23} as the interference is neutralized. Then Rx_2 provides the combined message $W_{23} + W_{21}$ to Rx_3. As Rx_3 receives $W_{31} + W_{23} - W_{12}$, it adds the known interference $W_{21} + W_{12}$ and substracts $W_{23} + W_{21}$ to finally decode W_{31}. All dedicated messages are decodable.

3.7 Summary

In this chapter, we have presented the three classical unidirectional communication channels, the multiple-access, broadcast and interference channel, for an introductory exposition of the cyclic polynomial channel model. Accordingly, three different and elementary types of *separability conditions* have been defined to analyze the feasibility of these channels: the multiple-access interference conditions, the intra-user interference conditions, and the inter-user interference conditions. Those conditions strongly depend on the interaction of dedicated and interfering messages between multiple users, and also on the number of incident and outgoing subchannels. Given the fulfillment of these conditions, the corresponding Degrees-of-Freedom have been derived for each of these channels. We have shown that it suffices to use orthogonal multiple-access schemes and linear encoding and decoding schemes to achieve the upper bounds on the Degrees-of-Freedom. Then, we have proposed cyclic interference alignment schemes to achieve the upper bounds on the Degrees-of-Freedom for the 2-user X-channel and the K-user interference channel. The results are in accordance with the Degrees-of-Freedom of their counterpart Gaussian channels with single antennas.

In particular, we have observed a *complementary reciprocal symmetry property* of interference alignment in the 2-user X-channel. This property basically states that aligning signals from both transmitters at a receiver Tx_1 provides a particular interference pattern at receiver Tx_2, and vice versa, aligning the same signals at receiver Tx_1 provides the reciprocal interference pattern observed before at receiver Tx_1. This property is shown to be significant for the optimal interference alignment scheme of the 2-user X-channel with arbitrary message lengths. To the best of our knowledge, such a property has not been discussed in the literature yet.

Furthermore, we have related the separability conditions to the common eigenvector problem of separate subspaces in MIMO interference alignment. We have observed that a set of feasibility conditions for the cyclic polynomial K-user interference channel can be formulated in analogy to the feasibility conditions for the MIMO channel model.

Moreover, we have shown that perfect cyclic interference alignment is infeasible in the 3-user X-network and hence also for K-user X-networks with $K \geq 3$ user-pairs. In order to counteract the infeasibility we allowed a minimal manipulation: We approached this infeasibility problem by providing a minimal number of cognitive messages to a subset of users via limited backhaul networks. Therein, we observed a duality relationship between cyclic interference alignment and cancellation with a backhaul network at the receivers versus cyclic interference neutralization with a backhaul network at the transmitters.

Altogether, the present chapter has provided Degrees-of-Freedom-achieving for several unidirectional multi-user networks in terms of the cyclic polynomial channel model using multiple-access, linear encoding and decoding and cyclic interference alignment schemes.

4 Multi-User Two-Hop and Two-Way Relay-Networks

In this chapter, we extend our focus to the CPCM for a unidirectional two-hop interference network with two transmitters, two relays and two receivers:

The $2 \times 2 \times 2$ **relay-interference channel**[1],

and introduce the concept of *cyclic interference neutralization* (IN). Furthermore, we extend the CPCM and cyclic IN to two-way (bidirectional) communications on:

The **two-way** $2 \times 2 \times 2$ **relay-interference channel**[2].

We also consider the following generalization of a multi-user two-way relay channel:

The **cascaded two-way relay channel**[3].

4.1 Cyclic Interference Neutralization

So far, we observed that IA in single-hop multi-user networks is necessary to achieve the maximal number of DoF. For communication scenarios with weak direct links between the dedicated users, however, the use of relays is beneficial to boost the data rate. We consider relays which forward their received signals to the dedicated destination within two hops and we assume that there is no direct link between the users. We assume for the sake of simplicity that relays have full-duplex capability and can perfectly cancel the loop-back self-interference. The considered relays are only intended to support communication, so that they do not desire to convey their own messages to other users.

If signals are forwarded in cooperative multi-hop networks with multiple interjacent relays, e. g., the $2 \times 2 \times 2$ *relay-interference channel* as depicted in Figure 4.1, IA can be extended to exploit *interference neutralization (IN)*. Such a $2 \times 2 \times 2$ relay-interference channel comprises two sources $\mathrm{Tx}_1, \mathrm{Tx}_2$, two parallel non-interfering interjacent relays $\mathrm{R}_1, \mathrm{R}_2$, and two destinations $\mathrm{Rx}_1, \mathrm{Rx}_2$. A source Tx_i intends to communicate a message W_i to its dedicated destination Rx_i with the aid of both relays R_1 and R_2 within two hops. In the first hop, the signals from Tx_1 and Tx_2 interfere at both relays. In the second hop, the relays forward and re-encode their superimposed messages which will also interfere at the destinations Rx_1 and Rx_2.

[1]Parts of this work have been published in [28].
[2]Parts of this work have been published in [29].
[3]Parts of this work have been published in [31].

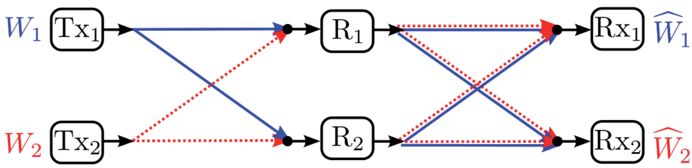


Figure 4.1: Neutralization on the $2 \times 2 \times 2$ relay-interference channel: Blue solid lines describe the signal paths of message W_1. Red dotted lines describe the signal paths of message W_2.

IN is a cooperative signalling scheme at both the sources and relays such that the interfering signals at undesired destinations are literally 'erased over the air'. In other words, the two incident paths of the corresponding interference signals at a destination Rx_i must exactly coincide and erase each other, while the dedicated signals remain (cf. Figure 4.1). As both relays receive all signals from both transmitters, they forward theses signals such that only the interference *neutralizes* itself at each receivers. This approach generates an effective channel as if there were no interference present between the sources and their dedicated receivers. Thus, the communication from a source Tx_i to its destination Rx_i becomes interference-free over two hops. Hence, the cut-set upper bounds on the DoF of the relay-interference channel are achievable.

To our knowledge, the original idea of IN has been derived and discussed in [47] in terms of the conceptual LDCM at first. A generalization to a *Z-chain relay-interference network*, i. e., a K-fold concatenation of Z-shaped relay-interference channels, is performed in [48].

For a MIMO 2×2×2 relay-interference channel with time-varying channel coefficients, the work [63] introduces a closely related *aligned IN* scheme. In [65], an IN scheme on a $2 \times 2 \times 2$ relay-interference channel with interfering relays is investigated. The approximate ergodic capacity of a $2 \times 2 \times 2$ relay-interference channel based on IN and ergodic IA is discussed in [66]. A finite-field $2 \times 2 \times 2$ relay-interference channel is considered in [56]. The authors of [67] provide a more generalized IN scheme by *aligned interference diagonalization* on a $K \times K \times K$ relay-interference channel to achieve the corresponding cut-set upper bound.

In the following, we begin with the investigation of the (unidirectional) $2 \times 2 \times 2$ relay-interference channel in terms of the CPCM. We apply a corresponding cyclic IN scheme that asymptotically achieves its maximal number of DoF.

4.1.1 Cyclic Polynomial $2 \times 2 \times 2$ Relay-Interference Channel

The sets of user-indices and relay-indices are $\mathcal{K}_{\text{Rx}} = \mathcal{K}_{\text{Tx}} = \mathcal{K}_{\text{R}} = \mathcal{K} = \{1, 2\}$. Each source Tx_i desires to communicate a message $\boldsymbol{w}_i$ to a dedicated destination Rx_i for $i \in \mathcal{K}$. The message from Tx_i is represented by a vector n submessages:

$$\boldsymbol{w}_i = (W_i^{[0]}, W_i^{[1]}, ..., W_i^{[n-1]}). \tag{4.1}$$

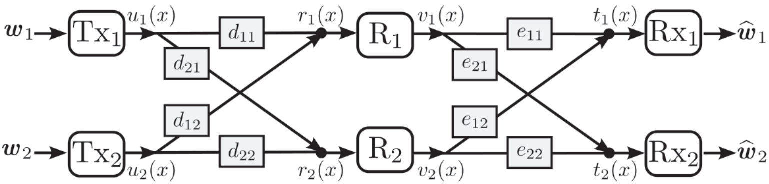


Figure 4.2: The polynomial $2 \times 2 \times 2$ relay-interference channel with channel matrix $\boldsymbol{D} = (d_{ji})^{2\times 2}$ between transmitters Tx_i and relays R_j and channel matrix $\boldsymbol{E} = (e_{ji})^{2\times 2}$ between relays R_i and destinations Rx_j.

The messaging matrix (for end-to-end communication from Tx_1, Tx_2 to Rx_1, Rx_2) is:

$$\boldsymbol{M} = n\boldsymbol{I}_{2\times 2} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}. \tag{4.2}$$

There is no direct link between the sources and destinations. Thus, the communication is performed over two hops by the aid of two full-duplex relays R_1 and R_2. The relays receive signals of the first hop at time-instant t. Concurrently, the relays apply a causal relaying function to forward their received signals of the previous first hop of time-instant $t-1$.

The block of transmitted signals $u_i(x)$ in one hop has n dimensions. The subchannel of the first hop is denoted by $\boldsymbol{D} = (d_{ji})_{1\le i,j\le 2}$ and the subchannel of the second hop by $\boldsymbol{E} = (e_{ji})_{1\le i,j\le 2}$ with $d_{ji}, e_{ji} \in \mathcal{D}$. Furthermore, the offset exponents are denoted by δ_{ji}, $\eta_{ji} \in \mathbb{N}$, so that $d_{ji} = x^{\delta_{ji}}$ and $e_{ji} = x^{\eta_{ji}}$.

The sources Tx_i transmit n submessages in polynomials of n dimensions:

$$u_i(x) = \textstyle\sum_{k=0}^{n-1} u_i^{[k]} x^k. \tag{4.3}$$

The received polynomial at a relay R_j is a cyclically shifted superposition of the transmitted polynomials from the sources S_i:

$$r_j(x) \equiv \textstyle\sum_{i\in\mathcal{K}} d_{ji} u_i(x) \mod (x^n - 1). \tag{4.4}$$

Both relays R_i map and encode their received polynomials $r_i(x)$ to construct the forwarded polynomials $v_i(x)$. This mapping may involve a *permutation of coefficients*, a *change of sign*, and even *discarding some specified coefficients* of the received polynomials. After the signals are forwarded from R_1 and R_2, the received polynomial at destination Rx_j yields:

$$t_j(x) \equiv \textstyle\sum_i e_{ji} v_i(x) \mod (x^n - 1). \tag{4.5}$$

The whole model of the $2 \times 2 \times 2$ relay-interference channel is also illustrated in Figure 4.2. The receivers Rx_i can only resolve and decode their desired messages, if the coefficients of the received polynomials $t_j(x)$ are linear interference-free.

The transmission vector of the first hop is denoted by $\boldsymbol{u} = (u_1(x), u_2(x))$ and the received vector by $\boldsymbol{r} = (r_1(x), r_2(x))$ in the vectorized notation. Accordingly, the transmission vector for the second hop is $\boldsymbol{v} = (v_1(x), v_2(x))$ and the received vector is $\boldsymbol{t} = (t_1(x), t_2(x))$. Then, the transfer functions of both hops can be compactly expressed by:

$$\boldsymbol{r}^\mathsf{T} = \boldsymbol{D}\boldsymbol{u}^\mathsf{T} \bmod (x^n - 1), \tag{4.6}$$
$$\boldsymbol{t}^\mathsf{T} = \boldsymbol{E}\boldsymbol{v}^\mathsf{T} \bmod (x^n - 1), \tag{4.7}$$

where the modulo-operation is component-wise.

In the following, we consider the upper bounds on the DoF of the $2 \times 2 \times 2$ relay-interference channel and the proposed achievable scheme.

4.1.2 Cut-Set Upper Bounds

The capacity is limited by the min-cut upper bounds [63] which are valid for both the CPCM and the time-variant MIMO channel model. Thus, presuming that each message is received interference-free at its dedicated receiver, each user-pair would achieve the capacity of the corresponding point-to-point link. In total there is a maximum number of $M = 2n$ interference-free submessages, i. e., n submessages per user to be conveyed over n dimensions so that the maximal data rate is upper bounded by the cut-set bound of 2 DoF.

4.1.3 Achievability

Our main goal is to formulate an interference-free transmission for all messages from each source to each dedicated destination. Note that instead of decoding single messages at both relays, only functions of superimposed messages are decoded in a network-coded fashion. These functions of superimposed messages are forwarded to the destinations using a proper relaying function. In the following, let the superscript indices in squared brackets be reduced by modulo n for notational convenience.

1) *First hop*: Each of the n submessages $W_i^{[k]}$, $k = 0, ..., n-1$, from source Tx_i is allocated to the corresponding dimension at offset x^k. The transmitted polynomial from source Tx_i yields:

$$u_i(x) = \textstyle\sum_{k=0}^{n-1} W_i^{[k]} x^k. \tag{4.8}$$

As given by (4.4), the relays R_j receive the following superposition of submessages per dimension:

$$r_j^{[k]} = W_1^{[k-\delta_{j1}]} + W_2^{[k-\delta_{j2}]}. \tag{4.9}$$

2) *Second hop*: The two relays forward their previously received polynomials by $v_1(x) = x^{\gamma_1} r_1(x) \bmod (x^n - 1)$ and $v_2(x) = -x^{\gamma_2} r_2(x) \bmod (x^n - 1)$, using the offset parameters $\gamma_1, \gamma_2 \in \{0, ..., n-1\}$, respectively. Using (4.5) and (4.9), the destinations Rx_1 and Rx_2 receive four submessages per dimension:

$$t_1^{[k]} = W_1^{[k-\delta_{11}-\gamma_1-\eta_{11}]} + W_2^{[k-\delta_{12}-\gamma_1-\eta_{11}]} - W_1^{[k-\delta_{21}-\gamma_2-\eta_{12}]} - W_2^{[k-\delta_{22}-\gamma_2-\eta_{12}]}, \tag{4.10}$$
$$t_2^{[k]} = W_1^{[k-\delta_{11}-\gamma_1-\eta_{21}]} + W_2^{[k-\delta_{12}-\gamma_1-\eta_{21}]} - W_1^{[k-\delta_{21}-\gamma_2-\eta_{22}]} - W_2^{[k-\delta_{22}-\gamma_2-\eta_{22}]}. \tag{4.11}$$

At both destinations, the dedicated submessages are superimposed by interference. The concept of IN is translated to aligning and combining two identical inter-user interference signals with complementary signs within the same dimension k, so that their sum is zero. To suppress the inter-user interference at both destinations, these two *interference-neutralization conditions* must hold:

$$\delta_{12} + \gamma_1 + \eta_{11} \equiv \delta_{22} + \gamma_2 + \eta_{12} \pmod{n}, \tag{4.12}$$
$$\delta_{11} + \gamma_1 + \eta_{21} \equiv \delta_{21} + \gamma_2 + \eta_{22} \pmod{n}. \tag{4.13}$$

In other words, the inter-user interference is aligned and *neutralized* over two hops along the two possible paths to the non-dedicated receiver.

On the other hand, we must also ensure that the desired signals remain intact and are not neutralized, i. e., the following *no-signal-neutralization conditions* must hold:

$$\delta_{11} + \gamma_1 + \eta_{11} \not\equiv \delta_{21} + \gamma_2 + \eta_{12} \pmod{n}, \tag{4.14}$$
$$\delta_{12} + \gamma_1 + \eta_{21} \not\equiv \delta_{22} + \gamma_2 + \eta_{22} \pmod{n}. \tag{4.15}$$

Let $\boldsymbol{\Gamma} = \mathsf{diag}(x^{\gamma_1}, -x^{\gamma_2})$ denote the relaying function as performed by both relays. The above conditions (4.12) to (4.15) indicate that the matrix product $\boldsymbol{E\Gamma D}$ must be a diagonal matrix of full rank, similar to the expression in [66, Section IV]. If these conditions are satisfied, the superposition of submessages given by (4.10) and (4.11) is reduced to:

$$t_1^{[k]} = W_1^{[k-\delta_{11}-\gamma_1-\eta_{11}]} - W_1^{[k-\delta_{21}-\gamma_2-\eta_{12}]}, \tag{4.16}$$
$$t_2^{[k]} = W_2^{[k-\delta_{12}-\gamma_1-\eta_{21}]} - W_2^{[k-\delta_{22}-\gamma_2-\eta_{22}]}. \tag{4.17}$$

The superposition of desired submessages received at destination Rx_j as given by (4.16) and (4.17) is compactly expressed by:

$$t_j(x) = (\boldsymbol{X}\boldsymbol{C}_j)\boldsymbol{w}_j^{\mathsf{T}}, \tag{4.18}$$

for $\boldsymbol{X} = \mathsf{diag}(x^0, x^1, x^2, ..., x^{n-1})$ and the $n \times n$ coefficient matrix $\boldsymbol{C}_j = (c_{j,lm})_{0 \le l,m \le n-1}$ with:

$$c_{j,lm} = \begin{cases} 1 & \text{, if } m - l \equiv \delta_{1j} + \gamma_1 + \eta_{j1} \pmod{n}, \\ -1 & \text{, if } m - l \equiv \delta_{2j} + \gamma_2 + \eta_{j2} \pmod{n}, \\ 0 & \text{, else.} \end{cases} \tag{4.19}$$

The superimposed submessages of $\boldsymbol{w}_j$ are resolvable by a linear decoding scheme, if the condition $\mathsf{det}(\boldsymbol{C}_j) \neq 0$ holds.

Lemma 4.1. *A linear decoding scheme at destination* Rx_i *can not resolve* n *desired submessages* $W_i^{[k]}$ *from the received vector* $\boldsymbol{t}_i$ *for the given perfect interference neutralization scheme.*

Proof:
$\boldsymbol{C}_j$ corresponds to an $n \times n$ circulant matrix $\widetilde{\boldsymbol{C}}_j$ (cf. [68]) with entries $c_{j,lm} = \tilde{c}_{j,(m-l \bmod n)}$. Thus, we have n eigenvectors of $\boldsymbol{C}_j$, namely $\boldsymbol{v}_i = \frac{1}{\sqrt{n}}(1, w_i, w_i^2, ..., w_i^{n-1})^{\mathsf{T}}$

for the indices $i = 0, ..., n-1$ with the roots of unity $w_i = \exp(\frac{\jmath 2\pi i}{n})$ and the complex symbol $\jmath = \sqrt{-1}$. The n corresponding eigenvalues are $\lambda_i = \sum_{k=0}^{n-1} \tilde{c}_k w_i^k$. Let $\nu_{ji} = \delta_{ij} + \gamma_i + \eta_{ji} \pmod{n}$. The determinant $\mathsf{det}(\boldsymbol{C}_j)$ is computed by the multiplication of n eigenvalues:

$$\begin{aligned}\mathsf{det}(\boldsymbol{C}_j) &= \prod_{j=0}^{n-1} \lambda_j = \prod_{j=0}^{n-1} (w_j^{\nu_{j1}} - w_j^{\nu_{j2}}) \\ &= (1^{\nu_{j1}} - 1^{\nu_{j2}}) \cdot \prod_{j=1}^{n-1} (w_j^{\nu_{j1}} - w_j^{\nu_{j2}}) = 0. \end{aligned} \tag{4.20}$$

Thence, the messages $\boldsymbol{w}_j$ can not be linearly resolved. ∎

In other words, the conditions (4.12) to (4.15) are too strict for a perfect cyclic IN scheme with a total number of $2n$ submessages. Thence, we propose an *asymptotic IN scheme* for $2n-1$ submessages:
1) *First hop*: Let source Tx_1 transmit n submessages as in (4.8) and let Tx_2 transmit only $n-1$ submessages, discarding a single submessage $W_2^{[\tau]}$ with the parameter $\tau \in \{0, ..., n-1\}$:

$$u_1(x) = \sum_{k=0}^{n-1} W_1^{[k]} x^k, \tag{4.21}$$
$$u_2(x) = \sum_{k=0, k\neq\tau}^{n-1} W_2^{[k]} x^k. \tag{4.22}$$

Now, the $k = 0, ..., n-1$ received dimensions at relays R_j are:

$$r_j^{[k]} = \begin{cases} W_1^{[k-\delta_{j1}]}, & \text{if } k \equiv \tau + \delta_{j2}, \\ W_1^{[k-\delta_{j1}]} + W_2^{[k-\delta_{j2}]}, & \text{otherwise}. \end{cases} \tag{4.23}$$

2) *Second hop*: Relay R_1 forwards all signal in its n dimensions and R_2 forwards only $n-1$ of the n received dimensions. In particular, relay R_2 discards forwarding the dimension received at $k_2 \equiv \tau + \delta_{22} \pmod{n}$. One γ_1, γ_2 is arbitrarily chosen and the other is computed by (4.12). The transmitted polynomials are:

$$v_1(x) = x^{\gamma_1} r_1(x) \bmod (x^n - 1), \tag{4.24}$$
$$v_2(x) = -x^{\gamma_2} \sum_{k=0, k\neq k_2}^{n-1} r_2^{[k]} x^k \bmod (x^n - 1). \tag{4.25}$$

The received signals at Rx_1, Rx_2 correspond to (4.10), (4.11). The discarded messages for $\sigma_{ji} = \tau + \delta_{i2} + \gamma_i + \eta_{ji} \pmod{n}$ yield:

$$t_1^{[\sigma_{11}]} = W_1^{[\sigma_{11}-\delta_{11}-\gamma_1-\eta_{11}]} - W_1^{[\sigma_{11}-\delta_{21}-\gamma_2-\eta_{12}]} - W_2^{[\sigma_{11}-\delta_{22}-\gamma_2-\eta_{12}]}, \tag{4.26}$$
$$t_1^{[\sigma_{12}]} = W_1^{[\sigma_{12}-\delta_{11}-\gamma_1-\eta_{11}]} + W_2^{[\sigma_{12}-\delta_{12}-\gamma_1-\eta_{11}]}, \tag{4.27}$$
$$t_2^{[\sigma_{21}]} = W_1^{[\sigma_{21}-\delta_{11}-\gamma_1-\eta_{21}]} - W_1^{[\sigma_{21}-\delta_{21}-\gamma_2-\eta_{22}]} - W_2^{[\sigma_{21}-\delta_{22}-\gamma_2-\eta_{22}]}, \tag{4.28}$$
$$t_2^{[\sigma_{22}]} = W_1^{[\sigma_{22}-\delta_{11}-\gamma_1-\eta_{21}]} + W_2^{[\sigma_{22}-\delta_{12}-\gamma_1-\eta_{21}]}. \tag{4.29}$$

Theorem 4.2. *The asymptotic interference neutralization scheme achieves* $\frac{2n-1}{n}$ *DoF on the cyclic polynomial channel, if the interference-neutralization conditions* (4.12), (4.13) *and no-self-neutralization conditions* (4.14), (4.15) *hold.*

Proof:
For the given conditions, the received signals at Rx_1, Rx_2 further simplify to (4.16), (4.17) and to these special cases:

$$t_1^{[\sigma_{1j}]} = W_1^{[\sigma_{1j}-\delta_{11}-\gamma_1-\eta_{11}]}, \tag{4.30}$$
$$t_2^{[\sigma_{21}]} = -W_2^{[\sigma_{21}-\delta_{22}-\gamma_2-\eta_{22}]}, \tag{4.31}$$
$$t_2^{[\sigma_{22}]} = W_1^{[\sigma_{22}-\delta_{11}-\gamma_1-\eta_{21}]} + W_2^{[\sigma_{22}-\delta_{12}-\gamma_1-\eta_{21}]}. \tag{4.32}$$

Note that $\sigma_{11} \equiv \sigma_{12} \pmod n$ holds here. Furthermore, the conditions (4.12) to (4.15) imply a proper choice of γ_1 and γ_2. At destination Rx_1, the coefficient matrix $\boldsymbol{C}_1$ has almost the same structure as in (4.19). The exception is an additional zero-entry in $\boldsymbol{C}_1$ at row σ_{1j} and column $\sigma_{1j} - \delta_{21} - \gamma_2 - \eta_{12}$ as given by (4.30). By Laplace's formula, we can recursively expand the determinant of $\boldsymbol{C}_1$ along the rows with only one non-zero entry, i. e., row σ_{1j} in the first iteration. The determinant yields $\mathsf{det}(\boldsymbol{C}_1) = 1$ and each submessage dedicated for Rx_1 is linear decodable.
Destination Rx_2 discards row σ_{22} and column τ in $\boldsymbol{C}_2$ since it only needs to decode the remaining $n-1$ submessages and not submessage $W_2^{[\tau]}$. Furthermore, the interfering submessage $W_1^{[\sigma_{22}-\delta_{11}-\gamma_1-\eta_{21}]}$ in (4.32) is not neutralized anyway. Thus, we consider a reduced coefficient matrix $\widehat{\boldsymbol{C}}_2$ which is a corresponding $(n-1) \times (n-1)$ matrix of $\boldsymbol{C}_2$. $\widehat{\boldsymbol{C}}_2$ has a single row with only one non-zero entry at σ_{21} as given in (4.31). In analogy to $\boldsymbol{C}_1$, the determinant yields $\mathsf{det}(\widehat{\boldsymbol{C}}_2) = 1$, so that each submessage dedicated for Rx_2 is also linear decodable.
Altogether, a total number of $M = 2n - 1$ submessages is conveyed interference-free over $n \geq 2$ dimensions using cyclic IN and linear decoding. The asymptotic scheme achieves $\lim_{n\to\infty} \frac{2n-1}{n} = 2$ DoF in the limit. ■

Note that valid channel parameters for $n \geq 2$ do exist, e. g., $d_{12} = e_{12} = x^1$, $d_{11} = d_{21} = d_{22} = e_{11} = e_{21} = e_{22} = x^0$.

Corollary 4.3. *The conditions of Theorem 4.2 also imply that:*

(a) $\delta_{12} + \delta_{21} + \eta_{11} + \eta_{22} \equiv \delta_{11} + \delta_{22} + \eta_{12} + \eta_{21} \pmod n$,
$\delta_{12} + \delta_{21} + \eta_{12} + \eta_{21} \not\equiv \delta_{11} + \delta_{22} + \eta_{11} + \eta_{22} \pmod n$,

(b) $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod (x^n - 1)$, $\mathsf{det}(\boldsymbol{E}) \not\equiv 0 \bmod (x^n - 1)$,

(c) and $n \geq 2$ dimensions,

must hold for cyclic IN.

Proof:
(a) The first condition, is obtained by substituting (4.12) and (4.13) w. r. t. γ_1 or γ_2. The same is done in (4.14) and (4.15) for the second condition, respectively.

(b) Assuming $\mathsf{det}(\boldsymbol{D}) \equiv 0 \bmod (x^n - 1)$, yields $\delta_{11} + \delta_{22} \equiv \delta_{12} + \delta_{21} \pmod n$. Inserting this into the first condition of (a), it follows $\mathsf{det}(\boldsymbol{E}) \equiv 0 \bmod (x^n - 1)$. Further inserting $\delta_{11} + \delta_{22} \equiv \delta_{12} + \delta_{21} \pmod n$ and $\eta_{11} + \eta_{22} \equiv \eta_{12} + \eta_{21} \pmod n$ into the second condition of (a), leads to a contradiction.

(c) By assuming $n = 1$, $\mathsf{det}(\boldsymbol{D}) \equiv \mathsf{det}(\boldsymbol{E}) \equiv 0 \bmod (x-1)$ always holds. This also leads to a contradiction as shown in (b). ∎

An illustrative example of cyclic IN is given in Appendix B.

4.1.4 Aligned IN on MIMO $2 \times 2 \times 2$ Relay-Interference Channels

The proposed asymptotic cyclic IA scheme of the previous section is not restricted for the application on the CPCM. In this section, we elaborate how the scheme can be extended to a generalized version of the *aligned interference neutralization* (AIN) scheme of [63] for a time-varying $2 \times 2 \times 2$ relay-interference channel.

System Model

We briefly recapitulate the channel model of the time-varying $2\times2\times2$ relay-interference channel for the AIN scheme as given in [63]. The aforementioned MIMO channel model is depicted in Figure 4.3.

1) *First hop*: The subchannel from source Tx_i to relay R_j is characterized by a channel coefficient $F_{ji} \in \mathbb{C}$. The relay R_j receives a superposition of the signals transmitted by the two sources Rx_i plus additive i. i. d. Gaussian noise $Z_{\mathrm{R}_j}(t) \sim \mathcal{CN}(0,1)$:

$$Y_{\mathrm{R}_j}(t) = \textstyle\sum_{i=1}^{2} F_{ji}(t) X_{\mathrm{Tx}_i}(t) + Z_{\mathrm{R}_j}(t). \tag{4.33}$$

2) *Second hop*: The subchannel from relay R_i to destination Rx_j is characterized by a channel coefficient $G_{ji} \in \mathbb{C}$. The destinations Rx_j receive a superposition of the signals from the relays R_i plus additive i. i. d. Gaussian noise, $Z_{\mathrm{Rx}_j}(t) \sim \mathcal{CN}(0,1)$:

$$Y_{\mathrm{Rx}_j}(t) = \textstyle\sum_{i=1}^{2} G_{ji}(t) X_{\mathrm{R}_i}(t) + Z_{\mathrm{Rx}_j}(t). \tag{4.34}$$

All sources, relays and destinations are equipped with single antennas only. The channel coefficients are generic and assumed to be time-varying in each discrete time-instant t. The values of these coefficients are bounded between a non-zero minimum and a finite maximum. Since the channel coefficients are generic, the matrices have full rank almost surely and are invertible, accordingly.

An *n-symbol extension* over n time-slots, as also utilized in [3] and [63], provides diagonal channel matrices over time-varying channel coefficients:

$$\boldsymbol{F}_{ji}(t) = \mathsf{diag}(F_{ji}(nt+1), ..., F_{ji}(nt+n)), \tag{4.35}$$
$$\boldsymbol{G}_{ji}(t) = \mathsf{diag}(G_{ji}(nt+1), ..., G_{ji}(nt+n)). \tag{4.36}$$

From now on, the time-instant t is dropped for notational brevity. We obtain the following received signals at the relays R_j and at the destinations Rx_k, as also depicted in Figure 4.3:

$$\boldsymbol{Y}_{\mathrm{R}_j} = \textstyle\sum_{i=1}^{2} \boldsymbol{F}_{ji} \boldsymbol{X}_{\mathrm{Tx}_i} + \boldsymbol{Z}_{\mathrm{R}_j}, \tag{4.37}$$
$$\boldsymbol{Y}_{\mathrm{Rx}_k} = \textstyle\sum_{i=1}^{2} \boldsymbol{G}_{ki} \boldsymbol{X}_{\mathrm{R}_i} + \boldsymbol{Z}_{\mathrm{Rx}_k}. \tag{4.38}$$

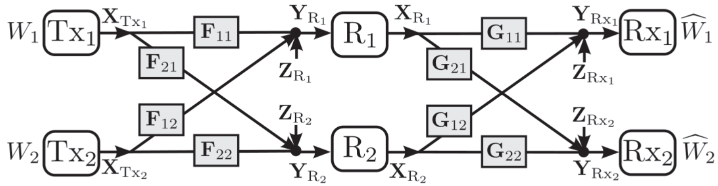


Figure 4.3: The channel model of [63] for the MIMO relay-interference channel with diagonal channel matrices $\boldsymbol{F}_{ji}$ between sources Tx_i, relays R_j and diagonal channel matrices $\boldsymbol{G}_{ji}$ between relays R_i and destinations Rx_j for $i, j \in \{1, 2\}$.

$\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ are $n \times 1$ vectors, i.e., the n-symbol extensions of X, Y, Z. Sources and relays encode their messages into Gaussian codebooks of length n, with codeword symbols $w^{[k]}_{\mathrm{Tx}_i}$, $w^{[k]}_{\mathrm{R}_i}$, and use beam-forming vectors $\boldsymbol{v}^{[k]}_{\mathrm{Tx}_i}$ and $\boldsymbol{v}^{[k]}_{\mathrm{R}_i}$, each of $n \times 1$ dimensions, to transmit the codewords over the given channel. The transmitted signals[4] from the Tx_i and R_i are:

$$\boldsymbol{X}_{\mathrm{Tx}_i} = \sum_{k=0}^{n-1} \boldsymbol{v}^{[k]}_{\mathrm{Tx}_i} w^{[k]}_{\mathrm{Tx}_i}, \tag{4.39}$$
$$\boldsymbol{X}_{\mathrm{R}_i} = \sum_{k=0}^{n-1} \boldsymbol{v}^{[k]}_{\mathrm{R}_i} w^{[k]}_{\mathrm{R}_i}. \tag{4.40}$$

The average transmit power for each transmit vector is limited by P.

(Cyclic) Generalization of the Aligned Interference Neutralization Scheme

An explanatory example for AIN is given in [63, Section I-D] for a symbol extension of $n = 2$ symbols. A linear AIN scheme for general $n \geq 2$ is given in [63, Section III-A]. Therein, AIN discards the symbols $w^{[n-1]}_{\mathrm{Tx}_2}, w^{[n-1]}_{\mathrm{R}_2}$ and aligns the beam-forming vectors for $i = 0, \ldots, n-2$ by:

$$\boldsymbol{F}_{11}\boldsymbol{v}^{[i+1]}_{\mathrm{Tx}_1} = \boldsymbol{F}_{12}\boldsymbol{v}^{[i]}_{\mathrm{Tx}_2}, \tag{4.41}$$
$$\boldsymbol{F}_{21}\boldsymbol{v}^{[i]}_{\mathrm{Tx}_1} = \boldsymbol{F}_{22}\boldsymbol{v}^{[i]}_{\mathrm{Tx}_2}, \tag{4.42}$$
$$\boldsymbol{G}_{11}\boldsymbol{v}^{[i+1]}_{\mathrm{R}_1} = -\boldsymbol{G}_{12}\boldsymbol{v}^{[i]}_{\mathrm{R}_2}, \tag{4.43}$$
$$\boldsymbol{G}_{21}\boldsymbol{v}^{[i]}_{\mathrm{R}_1} = -\boldsymbol{G}_{22}\boldsymbol{v}^{[i]}_{\mathrm{R}_2}. \tag{4.44}$$

There, it is proven that $\frac{2n-1}{n}$ DoF are achievable on the relay-interference channel.

We now show how this scheme can be interpreted as a special case of Theorem 4.2 with the cyclic IN scheme.

1) First hop:

Source Rx_1 sends n and Rx_2 sends $n-1$ symbols $w^{[0]}_{\mathrm{Tx}_j}, \ldots, w^{[n-1]}_{\mathrm{Tx}_j}$, along beam-forming vectors $\boldsymbol{v}^{[0]}_{\mathrm{Tx}_j}, \ldots, \boldsymbol{v}^{[n-1]}_{\mathrm{Tx}_j}$, for $j = 1, 2$, discarding $w^{[\tau]}_{\mathrm{Tx}_2}$ along $\boldsymbol{v}^{[\tau]}_{\mathrm{Tx}_2}$, respectively. In order

[4] Our notation of indices slightly differs from [63].

to imitate the separate dimensions of the CPCM, the beam-forming vectors align at the relays R_1 and R_2, for $i = 0, ..., n-1$, as follows:

$$\boldsymbol{F}_{11}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i-\delta_{11}]} = \boldsymbol{F}_{12}\boldsymbol{v}_{\mathrm{Tx}_2}^{[i-\delta_{12}]}, \ i \not\equiv \tau + \delta_{12} \pmod{n}, \tag{4.45}$$

$$\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i-\delta_{21}]} = \boldsymbol{F}_{22}\boldsymbol{v}_{\mathrm{Tx}_2}^{[i-\delta_{22}]}, \ i \not\equiv \tau + \delta_{22} \pmod{n}. \tag{4.46}$$

First, we equivalently express (4.45) and (4.46) by:

$$\boldsymbol{v}_{\mathrm{Tx}_2}^{[i]} = \boldsymbol{F}_{12}^{-1}\boldsymbol{F}_{11}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i+\delta_{12}-\delta_{11}]}, \ i \not\equiv \tau \pmod{n}, \tag{4.47}$$

$$\boldsymbol{v}_{\mathrm{Tx}_2}^{[i]} = \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i+\delta_{22}-\delta_{21}]}, \ i \not\equiv \tau \pmod{n}, \tag{4.48}$$

in order to substitute $\boldsymbol{v}_{\mathrm{Tx}_2}^{[i]}$:

$$\boldsymbol{F}_{11}\boldsymbol{F}_{12}^{-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i+\delta_{12}-\delta_{11}]} = \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i+\delta_{22}-\delta_{21}]}, \ i \not\equiv \tau \pmod{n}.$$

Then, we resolve the resulting equation w. r. t. $\boldsymbol{v}_{\mathrm{Tx}_1}^{[i]}$:

$$\boldsymbol{v}_{\mathrm{Tx}_1}^{[i+\delta_{12}-\delta_{11}+\delta_{21}-\delta_{22}]} = \boldsymbol{F}_{11}^{-1}\boldsymbol{F}_{12}\boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i]}.$$

Now, we use the following variables for brevity:

$$\Delta_{\boldsymbol{D}} \equiv \delta_{12} - \delta_{11} + \delta_{21} - \delta_{22} \pmod{n}, \tag{4.49}$$

$$\tau_1 \equiv \tau + \delta_{12} - \delta_{11} \pmod{n}, \tag{4.50}$$

$$\boldsymbol{F} = \boldsymbol{F}_{11}^{-1}\boldsymbol{F}_{12}\boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}. \tag{4.51}$$

Using these variables, we obtain the simpler expression:

$$\boldsymbol{v}_{\mathrm{Tx}_1}^{[i+\Delta_{\boldsymbol{D}}]} = \boldsymbol{F}\boldsymbol{v}_{\mathrm{Tx}_1}^{[i]}. \tag{4.52}$$

We fix $\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]} = \boldsymbol{1}_{n\times 1}$ for some $\kappa \in \{0, \dots, n-1\}$. To compute the corresponding vector from Tx_2 aligned to $\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}$, we substitute $\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+\Delta_{\boldsymbol{D}}]}$ on the left-hand side with (4.48), and obtain:

$$\begin{aligned} \boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+\Delta_{\boldsymbol{D}}]} &= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+\Delta_{\boldsymbol{D}}+\delta_{22}-\delta_{21}]} \\ \Rightarrow \boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+\Delta_{\boldsymbol{D}}-\delta_{22}+\delta_{21}]} &= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{F}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}. \end{aligned}$$

Then, by substituting $\boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+\Delta_{\boldsymbol{D}}-\delta_{22}+\delta_{21}]} = \boldsymbol{F}_{12}^{-1}\boldsymbol{F}_{11}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+2\Delta_{\boldsymbol{D}}]}$ with (4.47), we obtain:

$$\begin{aligned} \boldsymbol{F}_{12}^{-1}\boldsymbol{F}_{11}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+2\Delta_{\boldsymbol{D}}]} &= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]} \\ \Rightarrow \boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+2\Delta_{\boldsymbol{D}}]} &= \boldsymbol{F}^2\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}. \end{aligned}$$

Repeating these steps for all $i = 1, ..., n-1$ vectors, yields:

$$\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+i\Delta_{\boldsymbol{D}}]} = \boldsymbol{F}^i\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}.$$

For the computation of each $\boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+i\Delta_{\boldsymbol{D}}-\delta_{22}+\delta_{21}]}$, we obtain:

$$\begin{aligned}\boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+i\Delta_{\boldsymbol{D}}-\delta_{22}+\delta_{21}]} &= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{F}^{i}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}\\ \Rightarrow \boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+i\Delta_{\boldsymbol{D}}-\delta_{12}+\delta_{11}]} &= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{F}^{i-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]},\end{aligned}$$

for general $i = 1, ..., n-1$. Since only diagonal matrices are involved here, the multiplication is commutative, providing:

$$\boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+i\Delta_{\boldsymbol{D}}-\delta_{12}+\delta_{11}]} = \boldsymbol{F}^{i-1}\boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}. \tag{4.53}$$

Hence, we have the following extension[5] for the expressions in [63, Eqs. (48), (49)]:

$$\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa+i\Delta_{\boldsymbol{D}}]} = \boldsymbol{F}^{i}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}, \tag{4.54}$$
$$\boldsymbol{v}_{\mathrm{Tx}_2}^{[\kappa+i\Delta_{\boldsymbol{D}}+\delta_{21}-\delta_{22}]} = \boldsymbol{F}^{i}\boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\kappa]}. \tag{4.55}$$

Since $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \mod (x^n - 1)$ is assumed for the CPCM, it is clear that also $\Delta_{\boldsymbol{D}} \not\equiv 0 \pmod n$ holds. The initial offset for the allocation of the vectors from Tx_1 is chosen as:

$$\tau_1 = \tau + \delta_{12} - \delta_{11}. \tag{4.56}$$

This offset is aligned at R_1, so that it exactly fits into the offset-gap which was intentionally left open by Tx_2 at offset τ. We consider the following composite matrix for the n transmission vectors from Tx_1:

$$\boldsymbol{B} = (\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, \boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1+\Delta_{\boldsymbol{D}}]}, \boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1+2\Delta_{\boldsymbol{D}}]}, ..., \boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1+(n-1)\Delta_{\boldsymbol{D}}]}) \tag{4.57}$$
$$= (\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, \boldsymbol{F}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, \boldsymbol{F}^2\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, ..., \boldsymbol{F}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}). \tag{4.58}$$

To ensure that all n vectors are allocated, n and $\Delta_{\boldsymbol{D}}$ must be relatively prime, i. e., $\mathsf{gcd}(n, \Delta_{\boldsymbol{D}}) = 1$. Then, $\Delta_{\boldsymbol{D}}$ is a generator of the abelian (cyclic) additive group $\mathbb{Z}_n$.

It is yet to show that these vectors are not colinear in n dimensions. Let B_m denote the m-th diagonal entry of $\boldsymbol{F}$, so that (4.57) can also be represented by an $n \times n$ Vandermonde matrix:

$$\boldsymbol{B} = \begin{pmatrix} 1 & B_1 & B_1^2 & \cdots & B_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & B_{n-1} & B_{n-1}^2 & \cdots & B_{n-1}^{n-1} \end{pmatrix}. \tag{4.59}$$

Its well-known determinant yields the Vandermonde polynomial:

$$\mathsf{det}(\boldsymbol{B}) = \prod\nolimits_{0 \le i < j \le n-1} (B_j - B_i) \neq 0, \tag{4.60}$$

since the B_m are distinct almost surely. Thus, the n beam-forming vectors in $\boldsymbol{B}$ are linear independent. The $n-1$ beam-forming vectors transmitted from Tx_2 are also

[5] The proposed IN scheme in [63, Equations (48), (49)] resembles the following choice of parameters: $\delta_{21} = \delta_{22} = \delta_{11} = 0, \delta_{12} = 1, \tau = 0 \;\Rightarrow \Delta_{\boldsymbol{D}} = \kappa = 1$.

linear independent by an analogous computation. We consider the following composite matrix $\boldsymbol{C}$ for the $n-1$ transmission vectors from Tx_2: (with $\boldsymbol{v}_{\mathrm{Tx}_2}^{[\tau_1-\delta_{12}+\delta_{11}+i\Delta_{\boldsymbol{D}}]} = \boldsymbol{v}_{\mathrm{Tx}_2}^{[\tau+i\Delta_{\boldsymbol{D}}]}$)

$$\boldsymbol{C} = (\boldsymbol{v}_{\mathrm{Tx}_2}^{[\tau+\Delta_{\boldsymbol{D}}]}, \boldsymbol{v}_{\mathrm{Tx}_2}^{[\tau+2\Delta_{\boldsymbol{D}}]}, \dots, \boldsymbol{v}_{\mathrm{Tx}_2}^{[\tau+(n-1)\Delta_{\boldsymbol{D}}]}) \tag{4.61}$$

$$= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}(\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, \boldsymbol{F}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, \boldsymbol{F}^2\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}, \dots, \boldsymbol{F}^{n-2}\boldsymbol{v}_{\mathrm{Tx}_1}^{[\tau_1]}). \tag{4.62}$$

Let C_m denote the m-th diagonal element of $\boldsymbol{C}$. The according representation as an $n \times n-1$ Vandermonde matrix is:

$$\boldsymbol{C} = \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\begin{pmatrix} 1 & C_1 & C_1^2 & \cdots & C_1^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & C_{n-1} & C_{n-1}^2 & \cdots & C_{n-1}^{n-2} \end{pmatrix}. \tag{4.63}$$

Likewise, its determinant is non-zero, almost surely, and the vectors are linear independent:

$$\det(\boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\boldsymbol{C}) = \det(\boldsymbol{F}_{22}^{-1}) \cdot \det(\boldsymbol{F}_{21}) \cdot \prod_{0\le i<j\le n-1}(C_j - C_i) \neq 0. \tag{4.64}$$

Using the given pre-coding scheme, the received signal $\boldsymbol{Y}_{\mathrm{R}_j}$ is:

$$\boldsymbol{Y}_{\mathrm{R}_j} = \boldsymbol{F}_{j1}\sum_{k=0}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[k]}w_{\mathrm{Tx}_1}^{[k]} + \boldsymbol{F}_{j2}\sum_{k'=0,\,k'\neq\tau}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_2}^{[k']}w_{\mathrm{Tx}_2}^{[k']} + \boldsymbol{Z}_{\mathrm{R}_j}. \tag{4.65}$$

Then, the received signal is filtered by $\boldsymbol{F}_{jj}^{-1}\boldsymbol{Y}_{\mathrm{R}_j}$:

$$\begin{aligned}
\boldsymbol{F}_{11}^{-1}\boldsymbol{Y}_{\mathrm{R}_1} &= \sum_{k=0}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[k]}w_{\mathrm{Tx}_1}^{[k]} + \boldsymbol{F}_{11}^{-1}\boldsymbol{F}_{12}\sum_{k'=0,\,k'\neq\tau}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_2}^{[k']}w_{\mathrm{Tx}_2}^{[k']} + \boldsymbol{F}_{11}^{-1}\boldsymbol{Z}_{\mathrm{R}_1} \\
&= \sum_{k=0}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[k]}w_{\mathrm{Tx}_1}^{[k]} + \sum_{k'=0,\,k'\neq\tau}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[k'-\delta_{11}+\delta_{12}]}w_{\mathrm{Tx}_2}^{[k']} + \boldsymbol{F}_{11}^{-1}\boldsymbol{Z}_{\mathrm{R}_1}, \\
\boldsymbol{F}_{22}^{-1}\boldsymbol{Y}_{\mathrm{R}_2} &= \boldsymbol{F}_{22}^{-1}\boldsymbol{F}_{21}\sum_{k=0}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_1}^{[k]}w_{\mathrm{Tx}_1}^{[k]} + \sum_{k'=0\,k'\neq\tau}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_2}^{[k']}w_{\mathrm{Tx}_2}^{[k']} + \boldsymbol{F}_{22}^{-1}\boldsymbol{Z}_{\mathrm{R}_2} \\
&= \sum_{k=0}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_2}^{[k-\delta_{22}+\delta_{21}]}w_{\mathrm{Tx}_1}^{[k]} + \sum_{k'=0\,k'\neq\tau}^{n-1}\boldsymbol{v}_{\mathrm{Tx}_2}^{[k']}w_{\mathrm{Tx}_2}^{[k']} + \boldsymbol{F}_{22}^{-1}\boldsymbol{Z}_{\mathrm{R}_2}.
\end{aligned}$$

Note that the filtered noise is negligible here due to the high SNR assumption. The resulting symbols are analogously ordered (and cyclically shifted) for each dimension as in the cyclic IN scheme proposed in Section 4.1.3.

2) Second hop:

An analogous construction to obtain the n separate dimensions is performed in the second hop. The relays amplify and forward their received signals from the previous hop. Furthermore, the forwarded symbols are also index-shifted by the offsets γ_1, γ_2 given in (4.24) and (4.25).

Relay R_1 sends n and R_2 sends a number of $n-1$ symbols $x_{\mathrm{R}_j}^{[0]}, \dots, x_{\mathrm{R}_j}^{[n-1]}$ along beam-forming vectors $\boldsymbol{v}_{\mathrm{R}_j}^{[0]}, \dots, \boldsymbol{v}_{\mathrm{R}_j}^{[n-1]}$. Relay R_2 discards to forward $x_{\mathrm{R}_2}^{[\tau+\delta_{22}+\gamma_2]}$ along $\boldsymbol{v}_{\mathrm{R}_2}^{[\tau+\delta_{22}+\gamma_2]}$. The beam-forming vectors are aligned at the destinations Rx_1 and Rx_2, for $i = 1, \dots, n-2$, as follows:

$$\begin{aligned}
\boldsymbol{G}_{11}\boldsymbol{v}_{\mathrm{R}_1}^{[i-\eta_{11}]} &= -\boldsymbol{G}_{12}\boldsymbol{v}_{\mathrm{R}_2}^{[i-\eta_{12}]},\ i \not\equiv \tau+\delta_{22}+\gamma_2+\eta_{12} \pmod{n}, \\
\boldsymbol{G}_{21}\boldsymbol{v}_{\mathrm{R}_1}^{[i-\eta_{21}]} &= -\boldsymbol{G}_{22}\boldsymbol{v}_{\mathrm{R}_2}^{[i-\eta_{22}]},\ i \not\equiv \tau+\delta_{22}+\gamma_2+\eta_{22} \pmod{n}.
\end{aligned}$$

The dependencies of the beam-forming vectors are similarly resolved w. r. t. $\boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2]} = \mathbf{1}_{n-1}$ as in the first hop, for $i = 0, ..., n-2$:

$$\boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2+i\Delta_{\boldsymbol{E}}]} = \boldsymbol{G}^i \boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2]}, \tag{4.66}$$

$$\boldsymbol{v}_{\mathrm{R}_2}^{[\tau_2+i\Delta_{\boldsymbol{E}}+\eta_{21}-\eta_{22}]} = -\boldsymbol{G}^i \boldsymbol{G}_{22}^{-1} \boldsymbol{G}_{21} \boldsymbol{v}_{\mathrm{R}_1}^{[\tau_2]}, \tag{4.67}$$

using the parameters:

$$\Delta_{\boldsymbol{E}} \equiv \eta_{12} - \eta_{11} + \eta_{21} - \eta_{22} \pmod n, \tag{4.68}$$

$$\tau_2 \equiv \tau + \delta_{22} + \gamma_2 + \eta_{12} - \eta_{11} \pmod n, \tag{4.69}$$

$$\boldsymbol{G} = (\boldsymbol{G}_{11}^{-1} \boldsymbol{G}_{12} \boldsymbol{G}_{22}^{-1} \boldsymbol{G}_{21}). \tag{4.70}$$

Note that (4.66) and (4.67) are basically analogous to (4.54) and (4.55).

Since $\mathsf{det}(\boldsymbol{E}) \not\equiv 0 \bmod (x^n - 1)$ is assumed, $\Delta_{\boldsymbol{E}} \not\equiv 0 \pmod n$ holds, as well. As in the first hop, n and $\Delta_{\boldsymbol{E}}$ must be coprime. Then, the linear independence of vectors can be shown analogously to the scheme given for the first hop. The received signals $\boldsymbol{Y}_{\mathrm{Rx}_j}$ are filtered with $\boldsymbol{G}_{jj}^{-1}$. The resulting symbols are comparably ordered as in (4.10) and (4.11) with the special cases of (4.30) to (4.32).

As a result, we can apply the cyclic IA framework for IN of Section 4.1.3 on the transmitted symbols, and achieve $\frac{2n-1}{n}$ DoF by Theorem 4.2.

The linear scheme of [63] expressed by (4.41) to (4.44) is translated to the CPCM using $n \in \mathbb{N}$ dimensions, parameter $\tau = n - 1$, and channel matrices $\boldsymbol{D}$ and $\boldsymbol{E}$ with the following entries, $d_{12} = e_{12} = x^1$, $d_{11} = d_{21} = d_{22} = e_{11} = e_{21} = e_{22} = x^0$.

4.2 Two-Way Cyclic Interference Neutralization

Now, we further generalize the concept of cyclic IN to a two-way version of the full-duplex $2 \times 2 \times 2$ relay-interference channel. The arrangement of users in this channel is similar to [69] and [70].

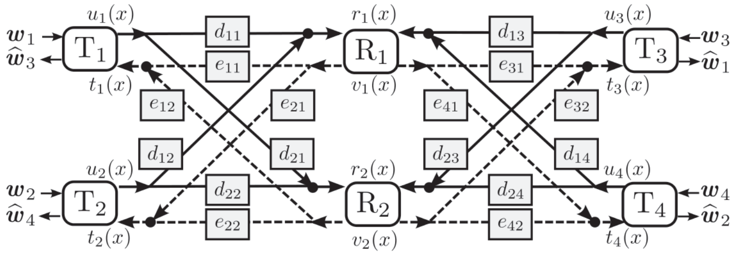


Figure 4.4: The $2 \times 2 \times 2$ full-duplex two-way relay-interference channel: Transceivers T_i transmit their signals $u_i(x)$ to relays R_j over the uplink channel matrix $\boldsymbol{D} = (d_{ji})_{1 \le j \le 2, 1 \le i \le 4}$ and the relays receive $r_j(x)$. Relays R_j forward signals $v_j(x)$, as functions of $r_j(x)$, over the downlink channel matrix $\boldsymbol{E} = (e_{ij})_{1 \le i \le 4, 1 \le j \le 2}$ to transceivers T_i, who receive the corresponding $t_i(x)$.

4.2.1 Cyclic Polynomial Two-Way $2 \times 2 \times 2$ Relay-Interference Channel

Such a channel comprises four full-duplex transceivers $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \mathrm{T}_4$, and two full-duplex relays $\mathrm{R}_1, \mathrm{R}_2$, as depicted in Figure 4.4. Each transceiver T_i transmits a message $\boldsymbol{w}_i$, $i \in \mathcal{K}_\mathrm{T}$. The user-pair $(\mathrm{T}_1, \mathrm{T}_3)$ desires to exchange messages $\boldsymbol{w}_1$ and $\boldsymbol{w}_3$ over the given channel, and the user-pair $(\mathrm{T}_2, \mathrm{T}_4)$ desires to exchange messages $\boldsymbol{w}_2$ and $\boldsymbol{w}_4$, respectively. There are no direct links between the transceivers and no direct links between the two relays. The messaging matrix (for end-to-end communication from all T_i to all T_j) is:

$$\boldsymbol{M} = n \begin{pmatrix} \boldsymbol{I}_{2\times2} & \boldsymbol{0}_{2\times2} \\ \boldsymbol{0}_{2\times2} & \boldsymbol{I}_{2\times2} \end{pmatrix} = \begin{pmatrix} 0 & n & 0 & 0 \\ n & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & n & 0 \end{pmatrix}. \tag{4.71}$$

Note that this channel can also be interpreted as a cognitive unidirectional $4 \times 2 \times 4$ relay-interference channel with 4 dedicated transmitter-receiver pairs with cognitive messages between pairs of Tx_i and Rx_i for each $i \in \mathcal{K}$. The related cognitive $3 \times 2 \times 3$ relay-interference channel is included by simply removing a dedicated transmitter-receiver pair, e. g., Tx_4 and Rx_2. Furthermore, the two-way $2 \times 2 \times 2$ relay-interference channel is a superposition of four (unidirectional) $2 \times 2 \times 2$ relay-interference channels:

1) The dedicated links are $T_1 \to T_3$ and $T_2 \to T_4$
2) The dedicated links are $T_3 \to T_1$ and $T_4 \to T_2$
3) The dedicated links are $T_1 \to T_3$ and $T_4 \to T_2$
4) The dedicated links are $T_3 \to T_1$ and $T_2 \to T_4$.

The *set of dedicated transmitter-indices* for a receiver T_i is denoted by $\mathcal{D}_i$, i. e., the singleton sets $\mathcal{D}_1 = \{3\}$, $\mathcal{D}_2 = \{4\}$, $\mathcal{D}_3 = \{1\}$ and $\mathcal{D}_4 = \{2\}$. We combine the indices of the two communicating user-pairs in the sets $\mathcal{G}_{13} = \{1,3\}$ and $\mathcal{G}_{24} = \{2,4\}$. The *set of interfering transmitter-indices* at a receiver R_i is denoted by $\mathcal{I}_i$, i. e., $\mathcal{I}_1 = \mathcal{I}_3 = \{2,4\}$ and $\mathcal{I}_2 = \mathcal{I}_4 = \{1,3\}$.

In both two-hop and two-way relay communication systems, the channel access is usually described by two different access phases: The *multiple-access phase* or *first hop* describes the communication from transceivers to relays and the *broadcast phase* or *second hop* describes the communication from relays to transceivers, accordingly. For multiple-relays, we will term these phases as the *uplink-phase* and the *downlink-phase*.

To model individual cyclic shifts for each transmitter-receiver link, the uplink (UL) from the four transceivers to the two relays is described by the *UL channel matrix* $\boldsymbol{D} = (d_{ji})_{1 \le j \le 2, 1 \le i \le 4}$ and the downlink (DL) from the relays is described by the *DL channel matrix* $\boldsymbol{E} = (e_{ij})_{1 \le i \le 4, 1 \le j \le 2}$ with $d_{ji}, e_{ji} \in \mathcal{D} = \{x^k \mid k \in \mathbb{N}\}$, respectively. We denote the offsets by δ_{ji}, $\eta_{ji} \in \mathbb{N}$, i. e., $d_{ji} = x^{\delta_{ji}}$ and $e_{ij} = x^{\eta_{ij}}$.

1) UL-phase: As in (4.3), the sources T_i, $i \in \mathcal{K}_\mathrm{T}$, map their messages $\boldsymbol{w}_i$ to a polynomial $u_i(x)$ with n dimensions:

$$u_i(x) = \sum\nolimits_{k=0}^{n-1} W_i^{[k]} x^k. \tag{4.72}$$

The polynomials $u_i(x)$ are transmitted to the relays R_j, $j \in \mathcal{K}_\mathrm{R}$, over the UL matrix $\boldsymbol{D}$ so that the two relays R_j receive a superposition of interfering polynomials:

$$r_j(x) = \sum_{i=1}^{4} d_{ji} u_i(x) \bmod (x^n - 1). \tag{4.73}$$

2) DL-phase: The relays R_j use a causal relaying function on their received polynomials $r_j(x)$. These polynomials are mapped to the polynomials $v_j(x)$. Then, the relays R_j forward their $v_j(x)$:

$$v_1(x) = x^{\gamma_1} r_1(x) \bmod (x^n - 1), \tag{4.74}$$
$$v_2(x) = -x^{\gamma_2} r_2(x) \bmod (x^n - 1), \tag{4.75}$$

to the four destinations T_i, $i \in \mathcal{K}$, over the DL matrix $\boldsymbol{E}$. A destination T_i receive the following superposition:

$$t_i(x) = \sum_{j=1}^{2} e_{ij} v_j(x) \bmod (x^n - 1). \tag{4.76}$$

The superposition of the three non-dedicated messages at destination T_i causes undesired interference. Only if all dedicated signals are received interference-free, the four destinations can decode their dedicated messages $\widehat{\boldsymbol{w}}_1, \widehat{\boldsymbol{w}}_2, \widehat{\boldsymbol{w}}_3$ and $\widehat{\boldsymbol{w}}_4$, respectively.

The submessages $\boldsymbol{w}_i$, which return from the relays to their original transceiver T_i during the DL-phase, are back-propagated self-interference [71]. Since the transceivers T_j know their own signals transmitted in the previous UL-phase, they can completely cancel their corresponding back-propagated self-interference.

The transmission vectors of the UL-phase are denoted by

$$\boldsymbol{u} = (u_1(x), u_2(x), u_3(x), u_4(x)), \tag{4.77}$$
$$\boldsymbol{r} = (r_1(x), r_2(x)), \tag{4.78}$$

respectively. In the DL-phase, we utilize the vectors:

$$\boldsymbol{v} = (v_1(x), v_2(x)),$$
$$\boldsymbol{t} = (t_1(x), t_2(x), t_3(x), t_4(x)).$$

Then, the transfer functions of the channel are:

$$\boldsymbol{r}^\mathsf{T} = \boldsymbol{D}\boldsymbol{u}^\mathsf{T} \bmod (x^n - 1), \tag{4.79}$$
$$\boldsymbol{t}^\mathsf{T} = \boldsymbol{E}\boldsymbol{v}^\mathsf{T} \bmod (x^n - 1), \tag{4.80}$$

where the modulo operation is taken component-wise.

4.2.2 Cut-Set Upper Bounds

For an interference-free transmission between four users with n dimensional signals, a total number of $M = 4n$ independent submessages must be decodable, i. e., n messages per user. A total number of exactly 4 DoF would be achieved for four interference-free subchannels. Such a result describes the cut-set upper bound on the DoF of the $2 \times 2 \times 2$ two-way relay-interference channel. In the following, we propose a combined cyclic SA and cyclic IN scheme that achieves the upper bound in the asymptotic limit for $n \to \infty$.

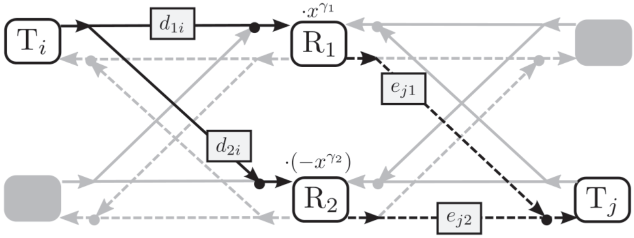


Figure 4.5: The interference-neutralization conditions in (4.81) demand that the identical signals transmitted by T_i coincide at each undesired transceivers T_j with complementary signs, so that interference is 'erased over the air'.

4.2.3 Achievability

We begin with the separability conditions of the two-way cyclic IN scheme by extending the previous separability conditions of the unidirectional $2 \times 2 \times 2$ relay-interference channel. First, we omit back-propagated self-interference, since it is cancelled at each transceiver. Thus, no additional self-interference condition comes up in this regard.

However, on the one hand, we demand that the *inter-user interference* caused by the two undesired transceivers in $\mathcal{I}_i$ is neutralized. The essential idea is again to combine two identical inter-user interference signals with complementary signs within the same dimension k, such that their sum vanishes. Thence, the *interference-neutralization conditions* for all interfering pairs, with $i \in \mathcal{K}_\mathrm{T}$, and $j \in \mathcal{I}_i$, are:

$$\delta_{1i} + \gamma_1 + \eta_{j1} \equiv \delta_{2i} + \gamma_2 + \eta_{j2} \pmod{n}. \tag{4.81}$$

This condition is also illustrated in Figure 4.5. The dedicated submessages on the other hand, may not be neutralized and must remain decodable. Accordingly, the *no-signal-neutralization conditions* for all dedicated pairs, with $i \in \mathcal{K}_\mathrm{T}$, and $j \in \mathcal{D}_i$, are:

$$\delta_{1i} + \gamma_1 + \eta_{j1} \not\equiv \delta_{2i} + \gamma_2 + \eta_{j2} \pmod{n}. \tag{4.82}$$

Altogether, assuming that the back-propagated self-interference signals are removed and the conditions (4.81) and (4.82) are fulfilled, the transceivers T_j only receive a superposition of two dedicated submessages per dimension k in the DL-phase:

$$t_j^{[k]} = W_i^{[k-\delta_{1i}-\gamma_1-\eta_{j1}]} - W_i^{[k-\delta_{2i}-\gamma_2-\eta_{j2}]}, j \in \mathcal{D}_i. \tag{4.83}$$

Using the vectorized notation as in 4.1.3:

$$\boldsymbol{w} = (W_i^{[0]}, W_i^{[1]}, \dots, W_i^{[n-1]}), \tag{4.84}$$

$$\boldsymbol{X} = \mathsf{diag}(x^0, x^1, \dots, x^{n-1}), \tag{4.85}$$

we can consider all n components of (4.83) together, yielding the received polynomial $t_j(x) = (\boldsymbol{X}\boldsymbol{C}_j)\boldsymbol{w}_j^\mathsf{T}$ with an $n \times n$ coefficient matrix $\boldsymbol{C}_j = (c_{j,lm})_{0 \le l,m \le n-1}$ with rows indexed by l and columns by m. For $i \in \mathcal{D}_j$, we obtain the following *circulant matrix* with two non-zero bands:

$$c_{j,lm} = \begin{cases} 1 & \text{, if } m - l \equiv \delta_{1i} + \gamma_1 + \eta_{j1} \pmod{n}, \\ -1 & \text{, if } m - l \equiv \delta_{2i} + \gamma_2 + \eta_{j2} \pmod{n}, \\ 0 & \text{, else.} \end{cases} \tag{4.86}$$

The received submessages are linear decodable, only if $\mathsf{det}(\boldsymbol{C}_j) \neq 0$ holds. In Lemma 4.1, we have already shown that cyclic IN is infeasible if all transmitting users allocate n submessages in n dimensions in a unidirectional $2 \times 2 \times 2$ relay-interference channel. Thus, as none of the four contained unidirectional relay-interference channels supports perfect cyclic IN for the case of n submessages per user, the two-way $2 \times 2 \times 2$ relay-interference channel does not support perfect cyclic IN either.

In order to enable cyclic IN with linear decoding nonetheless, we propose an extended *asymptotic* cyclic IN scheme that generalizes the cyclic IN scheme in Section 4.1.3 to the given $2 \times 2 \times 2$ two-way relay-interference channel:

1) UL-phase:

The transceivers $\mathrm{T}_1, \mathrm{T}_3$ transmit n submessages whereas $\mathrm{T}_2, \mathrm{T}_4$ only transmit $n-1$ submessages, discarding the submessages $W_2^{[\tau_2]}, W_4^{[\tau_4]}$ for $\tau_2, \tau_4 \in \{1, ..., n-1\}$:

$$u_i(x) = \textstyle\sum_{k=0}^{n-1} W_i^{[k]} x^k, \; i = 1, 3, \tag{4.87}$$

$$u_i(x) = \textstyle\sum_{k=0, k \neq \tau_i}^{n-1} W_i^{[k]} x^k, \; i = 2, 4. \tag{4.88}$$

The submessages received at the two relays per dimension k correspond to:

$$r_j^{[k]} = \textstyle\sum_{i=1}^{4} W_i^{[k-\delta_{ji}]}, \tag{4.89}$$

except for the following cases with $j \in \mathcal{K}_{\mathrm{R}}$, $m \in \{2, 4\}$:

$$r_j^{[\tau_m + \delta_{jm}]} = \textstyle\sum_{i=1, i \neq m}^{4} W_i^{[\tau_m + \delta_{jm} - \delta_{ji}]}. \tag{4.90}$$

We choose the parameters τ_2 and τ_4 such that:

$$\kappa_2 = \tau_2 + \delta_{22} \equiv \tau_4 + \delta_{24} \pmod{n} \tag{4.91}$$

holds, i. e., both the discarded submessages will affect exactly one dimension κ_2 at receiver R_2. Accordingly, we define $\kappa_{12} \equiv \tau_2 + \delta_{12} \pmod{n}$ and $\kappa_{14} \equiv \tau_4 + \delta_{14} \pmod{n}$ to describe the dimensions at R_1 that are affected by the discarded submessages from T_2 and T_4. Due to the interference-neutralization conditions, the dimensions of these discarded submessages are likewise aligned at R_1. To show this, we consider (4.81) for $i \in \{2, 4\}$ and $j = 1$:

$$\delta_{12} + \gamma_1 + \eta_{11} \equiv \delta_{22} + \gamma_2 + \eta_{12} \pmod{n},$$

$$\delta_{14} + \gamma_1 + \eta_{11} \equiv \delta_{24} + \gamma_2 + \eta_{12} \pmod{n}.$$

By substituting γ_1, we easily obtain:

$$\delta_{12} - \delta_{14} \equiv \delta_{22} - \delta_{24} \pmod{n}. \tag{4.92}$$

It follows from (4.91) and (4.92) that $\tau_2 + \delta_{12} \equiv \tau_4 + \delta_{14} \pmod{n}$ holds, i. e., we may set $\kappa_1 = \kappa_{12} \equiv \kappa_{14} \pmod{n}$.

2) DL-phase:

Relay R_1 forwards its received polynomial $r_1(x)$ according to:

$$v_1(x) = \quad x^{\gamma_1} r_1(x) \bmod (x^n - 1). \tag{4.93}$$

R_2 forwards only $n-1$ dimensions of the received polynomial $r_2(x)$ and discards $r_2^{[\kappa_2]}$:

$$v_2(x) = -x^{\gamma_2} \sum_{k=0,k\neq\kappa_2}^{n-1} r_2^{[k]} x^k \bmod (x^n - 1). \tag{4.94}$$

Let $\sigma_{ji} = \kappa_i + \gamma_i + \eta_{ji}$ for notational brevity. The received dimensions at the destinations T_j are as given in (4.83). The following cases result from the discarded coefficients, and from the self-interference cancellation for $j \in \mathcal{G}_{13}, i \in \mathcal{D}_j$:

$$\begin{aligned} t_j^{[\sigma_{j1}]} = \quad & W_i^{[\sigma_{j1}-\delta_{1i}-\gamma_1-\eta_{j1}]} - W_i^{[\sigma_{j1}-\delta_{2i}-\gamma_2-\eta_{j2}]} - W_2^{[\sigma_{j1}-\delta_{22}-\gamma_2-\eta_{j2}]} \\ & + W_4^{[\sigma_{j1}-\delta_{14}-\gamma_1-\eta_{j1}]} - W_4^{[\sigma_{j1}-\delta_{24}-\gamma_2-\eta_{j2}]}, \end{aligned} \tag{4.95}$$

$$\begin{aligned} t_j^{[\sigma_{j2}]} = \quad & W_i^{[\sigma_{j1}-\delta_{1i}-\gamma_1-\eta_{j1}]} - W_i^{[\sigma_{j2}-\delta_{2i}-\gamma_2-\eta_{j2}]} + W_2^{[\sigma_{j1}-\delta_{12}-\gamma_1-\eta_{j1}]} \\ & + W_4^{[\sigma_{j1}-\delta_{14}-\gamma_1-\eta_{j1}]} - W_4^{[\sigma_{j1}-\delta_{24}-\gamma_2-\eta_{j2}]}, \end{aligned} \tag{4.96}$$

and for $j \in \mathcal{G}_{24}, i \in \mathcal{D}_j$:

$$\begin{aligned} t_j^{[\sigma_{j1}]} = \quad & W_1^{[\sigma_{j1}-\delta_{11}-\gamma_1-\eta_{j1}]} - W_1^{[\sigma_{j1}-\delta_{21}-\gamma_2-\eta_{j2}]} - W_i^{[\sigma_{j1}-\delta_{2i}-\gamma_2-\eta_{j2}]} \\ & + W_3^{[\sigma_{j1}-\delta_{13}-\gamma_1-\eta_{j1}]} - W_3^{[\sigma_{j1}-\delta_{23}-\gamma_2-\eta_{j2}]}, \end{aligned} \tag{4.97}$$

$$t_j^{[\sigma_{j2}]} = \quad W_1^{[\sigma_{j2}-\delta_{11}-\gamma_1-\eta_{j1}]} + W_i^{[\sigma_{j2}-\delta_{1i}-\gamma_1-\eta_{j1}]} + W_3^{[\sigma_{j2}-\delta_{13}-\gamma_1-\eta_{j1}]}. \tag{4.98}$$

By further including the interference-neutralization conditions from (4.81), the equations (4.95) and (4.96) reduce to:

$$t_j^{[\sigma_{j1}]} = W_i^{[\sigma_{j1}-\delta_{1i}-\gamma_1-\eta_{j1}]}, i \neq j \in \mathcal{G}_{13}, \tag{4.99}$$

$$t_j^{[\sigma_{j2}]} = W_i^{[\sigma_{j2}-\delta_{2i}-\gamma_2-\eta_{j2}]}, i \neq j \in \mathcal{G}_{13}. \tag{4.100}$$

By definition of σ_{ji} and by condition (4.81), we observe that (4.97) and (4.98) coincide for each $j \in \mathcal{G}_{13}$. According simplifications also apply to (4.97) and (4.98):

$$t_j^{[\sigma_{j1}]} = -W_i^{[\sigma_{j1}-\delta_{2i}-\gamma_2-\eta_{j2}]}, \quad i \neq j \in \mathcal{G}_{24}, \tag{4.101}$$

$$t_j^{[\sigma_{j2}]} = \quad W_i^{[\sigma_{j2}-\delta_{1i}-\gamma_1-\eta_{j1}]} + W_1^{[\sigma_{j2}-\delta_{11}-\gamma_1-\eta_{j1}]} + W_3^{[\sigma_{j2}-\delta_{13}-\gamma_1-\eta_{j1}]}, \quad i \neq j \in \mathcal{G}_{24}. \tag{4.102}$$

The following theorem generalizes the (unidirectional) cyclic IN scheme in Theorem 4.2 of Section 4.1.3 to the present two-way case.

Theorem 4.4. *Asymptotic cyclic interference neutralization on the $2\times2\times2$ full-duplex two-way relay-interference channel achieves $\frac{4n-2}{n}$ DoF if all the following conditions hold:*

(a) back-propagated self-interference is cancelled at each T_i,

(b) *the separability conditions* (4.81) *and* (4.82) *hold,*

(c) *and the number of signalling dimensions is* $n \geq 2$.

Proof:
The n dedicated submessages received at T_j, $j \in \mathcal{G}_{13}$, are described by (4.83) and by the exceptions in (4.99) and (4.100). Now, the corresponding coefficient matrices $\boldsymbol{C}_j$ have almost the same structure as (4.86), except that the single entry in row σ_{j1} and column $\sigma_{j1} - \delta_{1i} - \gamma_1 - \eta_{j1}$ is zero. In this case, all n submessages at T_j with $j \in \mathcal{G}_{13}$ are decodable, since $\det(\boldsymbol{C}_j) = 1$ holds as in Theorem 4.2 of Section 4.1.3 for the unidirectional case.

The $n-1$ dedicated submessages at T_j, $j \in \mathcal{G}_{24}$, are also decodable. In this case, it suffices to consider a reduced $(n-1) \times (n-1)$ coefficient matrices $\widetilde{\boldsymbol{C}}_j$, since only $n-1$ submessages per transceiver must be decoded. Moreover, the interference in the remaining dimension is not neutralized anyway. In particular, the entry in row σ_{j2} and column with $W_i^{[\tau_i]}$, $j \in \mathcal{D}_i, j \neq i \in \mathcal{G}_{24}$ is discarded. Then, $\det(\widetilde{\boldsymbol{C}}_j) = 1$ for $j \in \mathcal{G}_{13}$, as analogously shown in Theorem 4.2.

By considering the derivation of (4.92), we observe that the proposed interference-neutralization conditions demand a particular symmetry of the considered channel. We subsume the symmetry for all analogous cases by the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}$:

$$\alpha_1 = \delta_{11} - \delta_{21} \equiv \delta_{13} - \delta_{23} \pmod{n}, \tag{4.103}$$
$$\alpha_2 = \delta_{12} - \delta_{22} \equiv \delta_{14} - \delta_{24} \pmod{n}, \tag{4.104}$$
$$\beta_1 = \eta_{11} - \eta_{12} \equiv \eta_{31} - \eta_{32} \pmod{n}, \tag{4.105}$$
$$\beta_2 = \eta_{21} - \eta_{22} \equiv \eta_{41} - \eta_{42} \pmod{n}. \tag{4.106}$$

Using this parameterization, the interference-neutralization conditions yield:

$$\alpha_1 + \gamma_1 \equiv \gamma_2 - \beta_2 \pmod{n}, \tag{4.107}$$
$$\alpha_2 + \gamma_1 \equiv \gamma_2 - \beta_1 \pmod{n}, \tag{4.108}$$

and the no-signal-neutralization conditions are:

$$\alpha_1 + \gamma_1 \not\equiv \gamma_2 - \beta_1 \pmod{n}, \tag{4.109}$$
$$\alpha_2 + \gamma_1 \not\equiv \gamma_2 - \beta_2 \pmod{n}. \tag{4.110}$$

Substituting (4.107) and (4.108) into (4.109) and (4.110) yields:

$$\alpha_1 - \alpha_2 \not\equiv \beta_2 - \beta_1 \pmod{n}. \tag{4.111}$$

in both cases. Valid matrices that fulfil these simplified conditions clearly exist, if $n \geq 2$ as demanded by condition (c).

Altogether, $4n-2$ submessages are conveyed interference-free over n dimensions. In the asymptotic limit, the cyclic IN scheme achieves $\lim_{n\to\infty} \frac{4n-2}{n} = 4$ DoF on the two-way $2 \times 2 \times 2$ relay-interference channel. ∎

4.2.4 Discussion

In contrast to Section 4.1 on the (unidirectional) 2×2 relay-interference channel, the present cyclic IN scheme can not be translated to a corresponding *aligned IN* scheme for a MIMO system with time-varying channel coefficients in the same manner.

The main problem, we encounter here is that the signals from the transceivers T_2 and T_4 aligned at R_1 are exactly the same ones aligned at R_2, since $\kappa_{12} \equiv \kappa_{14} \pmod{n}$ holds as given in Section 4.2.2. Although this problem is neither an issue for in the representation of the CPCM, nor for the closely related representation in the LDCM, this might be an overconstrained problem for the aligned IN framework [63] which is based on spatial IA and IN [3].

The related IN scheme given in [70] provides, among other results, that for all four terminals with M antennas each and N antennas at each relay, a total number of $\min(2M, 4N)$ DoF is achievable for $N \leq M$. The authors also mention the corresponding cut-set upper bounds of $\min(4M, 4N)$ DoF. However, it is shown that the cut-set upper bound of $4M$ DoF is indeed achievable, but under the condition that the relays are equipped with a greater number of $N > \frac{4}{3}M$ antennas by using the particular IN scheme provided in [70]. An interesting problem that remains to be solved is whether there is a feasible aligned IN scheme that is capable to achieve the cut-set upper bound of $4M$ DoF on the related Gaussian MIMO channel model with only $M = N$ antennas at the relays.

4.3 Cascaded Two-Way Relaying

We now consider two closely related two-way relay communication systems with full-duplex users and full-duplex intermediate relays. The elementary network problem of the well-known two-way relay channel [57] considers a pair of two users communicating via one relay, that is used to forward the bidirectional messages. Achievable rate regions for cooperation schemes are provided in [71] and [72] for two-way decode-and-forward or two-way compress-and-forward, respectively. An approximate capacity analysis of the two-way relay channel within 3 bits, involving the conceptual LDCM [9], is provided in [73].

Generalizations to $K \geq 2$ user-pairs communicating over a single relay are considered in [74] and [75]. A multi-hop scheme for a single user-pair over a a line of K two-way relays is presented in [76]. The given relaying scheme avoids the regeneration of back-propagated self-interference at the relays in particular. Another generalization of two-way relaying concerns multiple-unicast transmissions per user. For the case $K = 3$, this is called a Y-channel [77], [78], and [79]. In that case, each of the 3 users intends to transmit a single message to the other two users over a single two-way relay. Besides IA, the communication schemes achieving the upper bounds on the DoF also involve SA and the cancellation of back-propagated self-interference. Similar strategies are used for two-way relaying with network-coding.

Another kind of generalizations for two-way relaying concerns multiple-unicast transmissions per user. For the case $K = 3$, this is called a Y-channel as discussed in [77], [78], and [79]. In that case, each of the 3 users intends to transmit a single message to the other two users over a single two-way relay.

In this section, we investigate K cascaded concatenated two-way relay channels w. r. t. the CPCM as depicted in Figure 4.6. To the best of our knowledge, this kind of generalization of a two-way relay network has not been considered in the literature so far. More precisely, we consider two closely related systems with a number of K users and K, or $K-1$, relays forming either an open-loop line network or a closed-loop ring network, respectively. In contrast to the ordinary two-way relay channel, the transmissions of each user are involved in up to two neighbouring two-way relay channels. We provide an upper bound and an achievable scheme that is based on both SA and IA and the cancellation of back-propagated self-interference. Furthermore, we observe for the closed-loop case that switching the roles of users and relays yields a dual network with reciprocal channels.

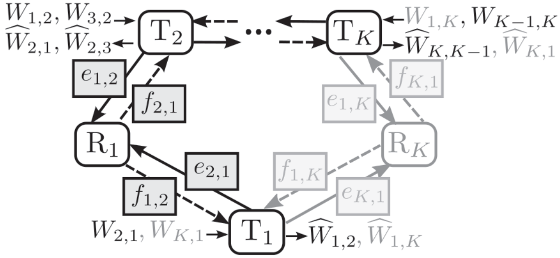


Figure 4.6: A cascaded two-way relay network: K users and K relays are alternately arranged. User T_i, is only connected to its predecessor T_{i-1} and its successor T_{i+1} via two-way relays R_{i-1} and R_i, respectively, for each $i = 1, ..., K$. The UL channel $\boldsymbol{E}$ is described by solid lines and the DL channel $\boldsymbol{F}$ by dashed lines. The open-loop case neglects Relay R_K, as indicated in grey.

4.3.1 Cyclic Polynomial Cascaded Two-Way Relay Networks

We now describe the two cascaded two-way relay networks with K-users in terms of the CPCM. We consider a full-duplex two-way relay network with K users T_i, $i \in \mathcal{K}_\mathrm{T}$, and K relays R_j, $j \in \mathcal{K}_\mathrm{R}$, for $K \geq 3$, as depicted in Figure 4.6. The sets of user-indices $\mathcal{K}_\mathrm{T}$ and relay-indices $\mathcal{K}_\mathrm{R}$ are both defined by $\mathcal{K}_\mathrm{T} = \mathcal{K}_\mathrm{R} = \mathcal{K}$.

Each user T_i, $i \in \mathcal{K}$, intends to convey one message $W_{i-1,i}$ to its predecessor T_{i-1} over relay R_{i-1} and another message $W_{i+1,i}$ to its successor T_{i+1} over relay R_i. Thus, there are $M = 2K$ independent messages W_{ji} in the system in total. For notational simplicity, indices $K+i$ correspond to i for a closed-loop of K users in a circular indexation. The messages $\widehat{W}_{ji}$ to be decoded at each T_j are denoted with a hat. To compactly describe the number of messages $m_{j,i} \in \mathbb{N}$ from T_i to T_j for each communication involved, we define a messaging matrix $\boldsymbol{M} = (m_{j,i})_{j,i \in \mathcal{K}}$ and set its entries to $m_{i,i+1} = m_{i,i-1} = 1$ and $m_{j,i} = 0$, else:

$$\boldsymbol{M} = \begin{pmatrix} \mathbf{0}_{1\times K-1} & 1 \\ \boldsymbol{I}_{K-1\times K-1} & \mathbf{0}_{K-1\times 1} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{K-1\times 1} & \boldsymbol{I}_{K-1\times K-1} \\ 1 & \mathbf{0}_{1\times K-1} \end{pmatrix}. \tag{4.112}$$

In other words, almost all elements are zero, except the side-diagonals and the top-right and the bottom-left entries. The total number of messages is $M = 2K$ for the given $K \times K$-matrix.

Next, we take a closer look at the signalling and the communication over the channel itself. The transmitted signal from T_i is a polynomial with messages W_{ji} for each intended receiver T_{i-1} and T_{i+1}:

$$u_i(x) \equiv W_{i-1,i} x^{p_{i-1,i}} + W_{i+1,i} x^{p_{i+1,i}} \mod (x^n - 1). \tag{4.113}$$

The parameter $p_{j,i} \in \{0, ..., n-1\}$ allocates the message W_{ji} to a particular offset within n dimensions.

The UL channel matrix describes all subchannels from the users T_i to the relays R_i. It is defined by $\boldsymbol{E} = (e_{j,i})_{j,i \in \mathcal{K}}$ and has the elements $e_{j,i} \in \mathcal{D}$ with the set of monomials $\mathcal{D} = \{x^k \mid k \in \mathbb{Z}^+\}$ describing the individual shifts from T_i to R_j. In $\boldsymbol{E}$, most elements are zero, except $e_{i,i+1}$ and $e_{i+1,i}$ for all $i \in \mathcal{K}$. The zero entries are similar to those in the messaging matrix $\boldsymbol{M}$. The DL channel matrix $\boldsymbol{F} = (f_{j,i})_{j,i \in \mathcal{K}}$ for subchannels from R_i to T_j is defined accordingly for $f_{j,i} \in \mathcal{D}$ with non-zero $f_{i,i+1}$ and $f_{i+1,i}$, for all $i \in \mathcal{K}$. In Figure 4.6, the UL subchannels are depicted by solid lines and the DL subchannels by dashed lines. The transfer function of the UL channel is the congruence:

$$\boldsymbol{r}^\top \equiv \boldsymbol{E}\boldsymbol{u}^\top \mod (x^n - 1), \tag{4.114}$$

with the $1 \times K$ input vector $\boldsymbol{u}$ and the $1 \times K$ output vector $\boldsymbol{r}$:

$$\boldsymbol{u} = (u_1(x), ..., u_K(x)), \tag{4.115}$$

$$\boldsymbol{r} = (r_1(x), ..., r_K(x)). \tag{4.116}$$

The modulo operation is taken element-wise. The signals $r_i(x)$ received at R_i are further processed as follows. For filtering certain dimensions from the received polynomial $r_i(x)$, we define the element-wise product of two polynomials $r_i(x)$ and $z_i(x)$, with maximal degree $n-1$, by:

$$r_i(x) \circ z_i(x) = \textstyle\sum_{k=0}^{n-1} r_i^{[k]} z_i^{[k]} x^k. \tag{4.117}$$

The yet unspecified *filter polynomial* $z_i(x)$ has filtering coefficients $z_i^{[k]} \in \{0, 1\}$ for all offsets $k \in \{0, 1, .., n-1\}$. The entries in $z_i(x)$ are chosen such that the dimensions with the undesired interference terms in $r_i(x)$ are removed (by multiplying zero) in the element-wise product $r_i(x) \circ z_i(x)$, whereas the dimensions of the dedicated signals in $r_i(x)$ remain unchanged (by multiplying one). The filtered polynomial in (4.117) may furthermore be cyclically shifted by x^{γ_i} with offset $\gamma_i \in \mathbb{N}$. The resulting polynomial after filtering and cyclic shifting is forwarded by R_i and denoted:

$$v_i(x) \equiv x^{\gamma_i}(r_i(x) \circ z_i(x)) \mod (x^n - 1). \tag{4.118}$$

The transfer function of the DL channel is the congruence:

$$\boldsymbol{t}^\top \equiv \boldsymbol{F}\boldsymbol{v}^\top \mod (x^n - 1), \tag{4.119}$$

for the $1 \times K$ input vector $\boldsymbol{v}$ and the $1 \times K$ output vector $\boldsymbol{t}$:

$$\boldsymbol{v} = (v_1(x), ..., v_K(x)), \tag{4.120}$$
$$\boldsymbol{t} = (t_1(x), ..., t_K(x)). \tag{4.121}$$

To ensure interference-free decodability of all dedicated signals, the *separability conditions* for $i \in \mathcal{K}$ as defined in Chapter 3 are adapted to this particular communication problem as follows:

> *Multiple-access interference conditions*: Dedicated messages to T_i transmitted by different sources T_{i-1} and T_{i+1} must be separable from each other at destination T_i:
>
> $$f_{i,i-1} x^{\gamma_{i-1}} e_{i,i-1} x^{p_{i,i-1}} \not\equiv f_{i,i+1} x^{\gamma_i} e_{i,i+1} x^{p_{i,i+1}} \bmod (x^n - 1). \tag{4.122}$$
>
> *Intra-user interference conditions*: Messages from the same source T_i, but dedicated for different destinations, must be distinct:
>
> $$x^{p_{i+1,i}} \not\equiv x^{p_{i-1,i}} \bmod (x^n - 1). \tag{4.123}$$
>
> *Inter-user interference conditions*: The dedicated messages from T_{i-1} to T_{i-2} and from T_{i+1} to T_{i+2} may not interfere with any of the dedicated signals for T_i:
>
> $$f_{i,i-1} x^{\gamma_{i-1}} e_{i,i-1} x^{p_{i-2,i-1}} \not\equiv f_{i,i+1} x^{\gamma_i} e_{i,i+1} x^{p_{i,i+1}} \bmod (x^n - 1), \tag{4.124}$$
> $$f_{i,i-1} x^{\gamma_{i-1}} e_{i,i-1} x^{p_{i,i-1}} \not\equiv f_{i,i+1} x^{\gamma_i} e_{i,i+1} x^{p_{i+2,i+1}} \bmod (x^n - 1). \tag{4.125}$$

4.3.2 Upper Bounds

For a fixed messaging matrix $\boldsymbol{M}$ the minimal number of dimensions necessary is bounded by (cf. the lower bound in (3.38)):

$$n \geq \max_{m_{ji}} \left(\sum_{i=1}^{K} m_{ji} + \sum_{j=1}^{K} m_{ji} - m_{ji} \right). \tag{4.126}$$

In other words, the lower bound on n is determined by the row j and column i of messaging matrix $\boldsymbol{M}$ that maximizes the sum in (4.126). For $K \geq 3$, the given messaging matrix $\boldsymbol{M}$ will always provide $n = 2$ for each $i, j \in \mathcal{K}$ and $M = 2K$, so that:

$$\mathrm{DoF} \leq \frac{M}{n} = \frac{2K}{2} = K. \tag{4.127}$$

Note that the case $K = 2$ corresponds to the elementary two-way relay channel which is also covered by this bound, since for $\boldsymbol{M} = \mathbf{1}_{2\times 2} - \boldsymbol{I}_{2\times 2}$, (4.126) provides $n = 1$ and hence the upper bound yields $\mathrm{DoF} \leq \frac{M}{n} = \frac{2}{1} = 2$.

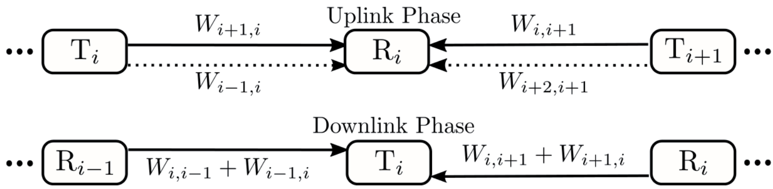


Figure 4.7: UL-phase: Signal alignment (solid arrows) of dedicated signals and interference alignment (dashed arrows) is performed at each relay R_i.
DL-phase: Superimposed dedicated messages at T_i are allocated by multiple-access and the self-interference is successively cancelled at T_i.

4.3.3 Achievability

In order to achieve the upper bound on the DoF as stated above, we propose a two-way relaying scheme including both network-coded cyclic SA and cyclic IA.

UL-phase: We consider the alignment scheme at the receiving relay R_i for all $i \in \mathcal{K}$. The two dedicated signals to be exchanged between T_i, T_{i+1} are aligned at R_i by SA:

$$e_{i+1,i} x^{p_{i+1,i}} \equiv e_{i,i+1} x^{p_{i,i+1}} \mod (x^n - 1). \tag{4.128}$$

The dedicated signals, superimposed at R_i, yield a network-coded messages, i. e., $W_{i+1,i} + W_{i,i+1}$. Likewise, the corresponding interference at R_i is aligned by IA:

$$e_{i+1,i} x^{p_{i-1,i}} \equiv e_{i,i+1} x^{p_{i+2,i+1}} \mod (x^n - 1). \tag{4.129}$$

DL-phase: We now consider the alignment scheme at the receiving user T_i for all $i \in \mathcal{K}$. The interference aligned in (4.129) is zero-forced by (4.118) with the zero-forcing particular polynomial:

$$z_i(x) = e_{i,i+1} x^{p_{i,i+1}} \equiv e_{i+1,i} x^{p_{i+1,i}} \mod (x^n - 1). \tag{4.130}$$

As a result, no inter-user interference is forwarded by R_i, so that the inter-user interference conditions (4.124) and (4.125) always hold. This is in contrast to the SA and IA schemes for the Y- channel in [77], where the inter-user interference is yet forwarded by the relays, but zero-forced at the destinations. The remaining dedicated network-coded signals from T_{i-1} to T_i are forwarded by R_{i-1} and the signals from T_{i+1} to T_i are forwarded by R_i.

This is a multicast problem, since each relay demands to convey a single albeit network-coded message to two transceivers simultaneously. The two forwarded signals received at T_i must yet satisfy the multiple-access conditions (4.122). To satisfy these conditions, a tuple of feasible parameters $(\gamma_1, ..., \gamma_K)$ is to be determined, if it exists.

T_i also receives back-propagated self-interference from R_{i-1} and from R_i. We apply self-interference cancellation at each T_i to remove the corresponding self-interference $W_{i-1,i}$ and $W_{i+1,i}$. Then, only the dedicated signals $W_{i,i-1}$ and $W_{i,i+1}$ remain, and can be decoded interference-free, if they satisfy the multiple-access conditions.

Case 1 - Open-Loop: Line Network

We first discuss the simpler case of the *open-loop line network*, discarding relay R_K w.l.o.g., so that the links between T_1 and T_K are severed and the parameters $p_{K,1}$, $p_{1,K}$ are neglected. In Figure 4.6, the affected links are highlighted in grey.

Theorem 4.5. *Cyclic IA achieves $K-1$ DoF on the given line network of K transceivers and $K-1$ relays a number of $n=2$ dimensions.*

Proof:
(a) *Necessity of $n \geq 2$:*
The multiple-access interference conditions (4.122) demand that two signals are decodable in the DL at each T_i, $i \in \mathcal{K}$, so that $n \geq 2$ is clearly necessary. The edge-transceivers T_1 and T_K are an exception and demand only $n \geq 1$, accordingly.

(b) *Sufficiency of Cyclic IA with $n=2$:*
UL-phase: We may fix the UL-transmission parameter $p_{2,1}$ at T_1 w.l.o.g. At T_2, we use the given SA in (4.128), to obtain the parameter $p_{1,2}$. As T_2 also has a dedicated message for transmission to T_3, we fix the parameter $p_{3,2}$ at T_2, satisfying the intra-user interference conditions in (4.123). At T_3, we use the given SA in (4.128) and the IA-condition in (4.129) to compute the parameters $p_{2,3}$ and $p_{4,3}$, respectively. Both satisfy the intra-user interference conditions in (4.123).

This is an iterative allocation procedure and it is analogously performed for each subsequent user $i \leq K-1$. At T_K, only the SA of (4.128) is needed to determine the parameters $p_{K-1,K}$ as $p_{1,K}=0$ produces no interference at T_K. There are no constraints on the channel matrix $\boldsymbol{E}$ of the UL and each parameter can be determined uniquely.

DL-phase: The zero-forcing polynomial of R_1 in (4.117) only contains the dimension of the dedicated signals, i. e., $z_1(x) \equiv e_{1,2}x^{p_{1,2}} \equiv e_{2,1}x^{p_{2,1}}$. The other coefficients in $z_1(x)$ are zero. As a result, the content of the complementary offset $e_{1,2}x^{p_{3,2}}$, with the interfering message $W_{2,1}$ in $r_1(x)$, is removed by $r_1(x) \circ z_1(x)$.

We can fix the parameter γ_1 for R_1 w. l. o. g. Due to the multiple-access interference conditions in (4.122), the signal from R_1 to T_2 may not align to the forwarded signal from R_2 to T_2, i. e., $x^{\gamma_1}f_{2,1} \not\equiv x^{\gamma_2}f_{2,3}$ must hold. A unique solution exists for γ_2 if $n=2$. Analogously, the γ_i for all other $3 \leq i \leq K-1$ can also be determined uniquely. There are no constraints on the channel matrix $\boldsymbol{F}$ of the DL either. The back-propagated self-interference is known at the transceivers and cancelled from the received signal.

As each T_i, with $2 \leq i \leq N-1$, sends two dedicated messages, and the edge-transceivers T_1 and T_K send one dedicated message each, a total number of $2K-2$ messages is conveyed interference-free over $n=2$ dimensions, yielding $\frac{2K-2}{2} = K-1$ DoF. ■

Case 2 - Closed-Loop: Ring Network

In this case, we consider the *closed-loop ring network* as depicted in Figure 4.6 (including the grey links) with an active relay R_K and non-zero messages between T_1 and T_K. We observe that this network imposes a constraint on $\boldsymbol{E}$ and $\boldsymbol{F}$.

Furthermore, UL and DL channels are called *reciprocal*, if:

$$e_{i,j}^{-1} \equiv f_{j,i} \bmod (x^n - 1), \tag{4.131}$$

holds for all $i \neq j \in \mathcal{K}$.

Theorem 4.6. *A cyclic IA/SA scheme achieves* DoF $\leq K$ *for* $n = 2$ *dimensions on the ring network with* K *transceiver and* K *relays, if the following UL- and DL-conditions for the channel matrices hold:*

$$\prod\nolimits_{i=1}^{K} e_{i+1,i}^{-1} e_{i,i+1} \equiv 1 \bmod (x^n - 1), \tag{4.132}$$

$$\prod\nolimits_{i=1}^{K} f_{i,i+1}^{-1} f_{i+1,i} \equiv 1 \bmod (x^n - 1). \tag{4.133}$$

Proof:
(a) *Necessity of* $n \geq 2$:
Cf. proof in Theorem 4.6(a), but without the edge-transceivers.

(b) *Necessity of the UL-condition* (4.132):
For a fixed $p_{1,K}$ in the SA-condition (4.128), we obtain:

$$x^{p_{1,K}} \equiv e_{1,K}^{-1} e_{K,1} x^{p_{K,1}}. \tag{4.134}$$

Then, we substitute the parameter $x^{p_{K,1}}$ by $x^{p_{K,1}} \equiv e_{2,1}^{-1} e_{1,2} x^{p_{3,2}}$ from the IA-condition (4.129) into (4.134):

$$x^{p_{1,K}} \equiv e_{1,K}^{-1} e_{K,1} e_{2,1}^{-1} e_{1,2} x^{p_{3,2}}.$$

In the next step, $x^{p_{3,2}}$ is substituted by using the SA-condition (4.128). By a simple unrolling of the dependencies with an alternating application of the IA and SA conditions in (4.128) and (4.129), for all $1 < i < K$, we obtain an additional product-term of $e_{i,i+1}^{-1} e_{i+1,i}$ for each i. In the last step, $x^{p_{1,K}}$ appears on both sides of the congruence. Clearly, the congruence is only true if the UL-condition (4.132) on $\boldsymbol{E}$ holds.

(c) *Sufficiency of Cyclic SA and IA in the UL with* $n = 2$:
The achievable scheme is analogous to Theorem 4.5 for the allocation of the parameter values $p_{i,i+1}$ and $p_{i+1,i}$ for all $i \in \{1, 2, ..., K-1\}$ for $n = 2$. A feasible solution for the two remaining parameters $p_{K,1}$ and $p_{1,K}$, satisfying the separability conditions, only exists if (4.132) holds.

(d) *Necessity of the DL-condition* (4.133):
For an arbitrary DL-matrix $\boldsymbol{F}$, a reciprocal UL-matrix $\boldsymbol{E}_{\text{rpr}}$ can be computed easily by (4.131). If $\boldsymbol{E}_{\text{rpr}}$ satisfies the UL-condition (4.132), then $\boldsymbol{F}$ will satisfy (4.133). As already given by parts (a), (b), and (c) of this proof, a feasible cyclic SA/IA scheme for the reciprocal UL with $\boldsymbol{E}_{\text{rpr}}$ exists. For the multicast transmission in the reciprocal DL channel $\boldsymbol{F}$, the superimposed dedicated signals are send back over exactly the same dimensions over which they have been received in the reciprocal UL with $\boldsymbol{E}_{\text{rpr}}$. As the intra-user interference conditions already hold in the reciprocal UL, the multiple-access conditions hold analogously in the DL by channel reciprocity. And conversely, if

(4.133) is violated, the parameters γ_i can not be determined uniquely for all $i = 1, ..., K$.

(e) *Sufficiency of Cyclic SA in the DL with* $n = 2$:
The inter-user interference conditions (4.124), (4.125) always hold since the inter-user interference is removed at each R_i by using the corresponding filtering polynomial. Now, the superimposed dedicated signals are multicast by each R_i and received at T_i and T_{i+1}. We may fix parameter γ_1. Again, $x^{\gamma_i} f_{i+1,i} \not\equiv x^{\gamma_i} f_{i+1,i+1}$ must hold for all γ_i with $i = 1, ..., K$, to ensure that the multiple-access conditions are not violated. This is valid for any $\boldsymbol{F}$ that satisfies (4.133). In the last step, the back-propagated self-interference is cancelled. Altogether, a total number of $\frac{2K}{2} = K$ DoF is achieved. ■

If the given UL- and DL-matrices are reciprocal, the conditions (4.132) and (4.133) become equivalent.

In case that at least one condition of Theorem 4.6 does not hold, the first straightforward approach is to demand at least $n = 3$ dimensions. Then, only a total number of $\frac{2K}{3}$ DoF is achievable. However a second and more efficient approach is to exploit the unconstrained Theorem 4.5 to achieve $K-1$ DoF by neglecting the dedicated messages between a single arbitrary pair $(\mathrm{T}_i, \mathrm{T}_{i+1})$ in both the UL and DL phases. For each transmission, a different pair of messages chosen by a scheduled round-robin scheme or chosen i.i.d. is neglected. The second approach clearly outperforms the first approach, since $\frac{2K}{3} < K - 1$ holds for $K > 3$.

4.3.4 User-Relay Duality

A *dual network* of the closed-loop case is physically the same network as the primary network with $\boldsymbol{E}$ and $\boldsymbol{F}$, but with former users operating as relays and former relays operating as users instead. Although the indexation of such a dual network is not unique for the closed-loop case of the cascaded two-way channel, it is obvious that the resulting dual network is indeed unique, since the equivalent indexation do not change the setup of dedicated messages between neighbouring users. W.l.o.g., we may relabel $\mathrm{R}_i \to \mathrm{T}_i$ and $\mathrm{T}_{i+1} \to \mathrm{R}_i$, i. e., the labels are cyclically right-shifted by one position. Let $\tilde{\boldsymbol{Z}}_K$ denote a particular circulant matrix with $\tilde{z}_{K-1} = 1$, and all other $\tilde{z}_j = 0$. The UL and DL matrices of the dual network are cyclically rotated versions $\boldsymbol{F}$ and $\boldsymbol{E}$ from the primary network:

$$\boldsymbol{E}_{\text{dual}} = \tilde{\boldsymbol{Z}}_K \boldsymbol{F}, \tag{4.135}$$

$$\boldsymbol{F}_{\text{dual}} = \boldsymbol{E} \tilde{\boldsymbol{Z}}_K^{\mathsf{T}}. \tag{4.136}$$

The corresponding dual UL and DL conditions yield:

$$\prod_{i=1}^{K} f_{i,i+1} f_{i+1,i}^{-1} \equiv 1 \bmod (x^n - 1), \tag{4.137}$$

$$\prod_{i=1}^{K} e_{i+1,i} e_{i,i+1}^{-1} \equiv 1 \bmod (x^n - 1). \tag{4.138}$$

Hence, $\boldsymbol{E}_{\text{dual}}$ satisfies the dual UL condition, if $\boldsymbol{F}$ satisfied the primary DL condition and $\boldsymbol{F}_{\text{dual}}$ satisfies the dual DL condition, if $\boldsymbol{E}$ satisfied the primary UL condition. Any cyclically right-shifted relabelling by an odd number of positions will yield the same dual conditions as given in (4.137) and (4.138). As a result, we observe that the following relationship:

User-Relay Duality: *If a cyclic SA and IA scheme is feasible on the primary network, it will also be feasible on any dual network.*

4.4 Summary

In this chapter, we have included full-duplex relays in order to extend the unidirectional communication schemes of the previous chapter and to enhance the achievable Degrees-of-Freedom.

First, we have considered an interference channel which was extended by two intermediate relays - a $2 \times 2 \times 2$ relay-interference channel. To achieve the cut-set upper bounds on the Degrees-of-Freedom, we have applied interference neutralization as introduced by Mohajer et al. in [47] in terms of the proposed cyclic polynomial channel model.

A particular gain of our provided problem description was the emphasis on the *interference-neutralization conditions* and *no-signal-neutralization conditions*. The approach has also allowed us to generalize the asymptotic MIMO interference alignment and neutralization scheme of Gou et al. in [63]. Afterwards, we have further equipped the users with perfect full-duplex capabilities and we have considered a two-way generalization of cyclic neutralization scheme. The given scheme has also incorporated the concept of *signal alignment* (dedicated signals are aligned), and of *back-propagated self-interference cancellation*.

In the last part of this chapter, we have focussed on a cascaded version of the well-known two-way relay channel. A user-relay duality and a unicast-multicast duality, which was exploited for the optimal DoF-achieving communication scheme, has been discussed in particular.

5 Multi-Way Communications with Three Users

In this chapter, we pursue the following goals:

> **Characterizing DoF-achieving and capacity-achieving schemes** for the 3-way channel and the 3-user Y-channel in terms of:
>
> - The **cyclic polynomial channel model** (CPCM),
> - the **linear deterministic channel model**[1] (LDCM), and
> - the **Gaussian MIMO channel model**[2] (GMCM).

Device-to-Device Communications: 3-Way versus Y-Channels

A natural communication scenario with multiple users is multi-way device-to-device (D2D) conferencing. Especially in wireless multi-user communication networks, this will involve multiple simultaneous transmissions causing interference that impairs the maximal achievable data rates per user. The D2D approach in [80], [81], and [82] is intended to increase the spectral efficiency of multi-way networks without the utilization of base stations for data transmission (except for low-rate top-level control mechanisms). As a countermeasure to deal with the impairment caused by interference, all transmission signals must be carefully designed so that mutual interference is minimized.

The 3-way channel (or Δ-channel) is an instance of a D2D-communication network with 3 users mutually communicating to each other, i. e., each user transmits 2 messages, one per receiver. We assume that all participating users employ perfect full-duplex devices. Note that the related *conferencing* 3 *-way channel* in [83] considers a single message per user that is multicast to the two other receivers.

A 3-user Y-channel as in [77], [78], and [79], is a closely related 3-way communication system, but it has an intermediate relay and no direct links between the users. Each transceiver T_i sends two messages to the relay R, and the relay forwards three network-coded messages back to the dedicated users. The DoF of the MIMO 3-user Y-channel with an M_i antennas per user T_i (There is an equal number of transmit and receive antennas, but a different number of antennas among different users) and M_{R} relay antennas are provided in [78]. Its generalization to a MIMO 4-user Y-channel is considered in [84] and [85]. In [79], the capacity region of the related 3-user Y-channel in the LDCM is derived.

[1] Parts of this work have been published in [34].
[2] Parts of this work have been published in [30] and [33].

The schemes proposed in the works on the Y-channel mainly rely on SA. Recall from the previous chapter, that SA is a particular adaptation of IA to multi-way relay networks, such that bidirectional dedicated signals (and not the interference signals) are aligned at the intermediate two-way relay.

Interestingly, the 3-way channel demands a combined scheme of various important communication techniques: IA, SA, IN, and backward decoding (BD).

Comparison of the GMCM and LDCM to the CPCM

Moreover, we have already pointed to several interesting properties that were also observed in the LDCM and the GMCM in the previous chapters: Up to now, we already enlightened the analogous representation of the upper bound on the DoF for $K_\mathrm{R} \times K_\mathrm{T}$ X-networks in Section 3.2, the problem of common eigenvectors in invariant subspaces discussed in Section 3.4.2 and 3.5.3, and the duality relationship of cyclic IAC and cyclic IN in Section 3.6.4, for instance. We will further elaborate conceptual connections between those different models within this chapter for the Y-channel and the 3-way channel.

5.1 Cyclic Interference Alignment for 3-Way and Y-Channels

We begin with the consideration of the 3-user Y-channel and the 3-way channel in terms of the CPCM. This approach serves us to provide a first-order understanding of the two aforementioned multi-way networks. Since the achievable scheme for the Y-channel is quite similar to the one in [79] for the LDCM, we would like to emphasize, that the provided scheme covers the case of arbitrary non-reciprocal shifts. Then, we investigate the Y-Δ product relationship as it occurs for the CPCM. To the best of our knowledge, such a product relationship has not been reported in terms of information theory for multi-user communications yet.

5.1.1 Cyclic Polynomial 3-User Y-Channel

In the Y-channel there are three users $\mathrm{T}_1, \mathrm{T}_2$, and T_3 linked to an intermediate full-duplex relay R. However, there is no direct link between the three transceivers. This arrangement is also depicted in Figure 5.1. The set of user-indices is $\mathcal{K}_\mathrm{T} = \mathcal{K} = \{1, 2, 3\}$. The number of submessages per dedicated user-pair (end-to-end) is given by the messaging matrix:

$$\boldsymbol{M} = \begin{pmatrix} 0 & m_{12} & m_{13} \\ m_{21} & 0 & m_{23} \\ m_{31} & m_{32} & 0 \end{pmatrix}. \tag{5.1}$$

Since there are no messages to the transmitting user itself, the main diagonal is assigned with zero values. The total number of submessages amounts to:

$$M = m_{12} + m_{21} + m_{13} + m_{31} + m_{23} + m_{32}.$$

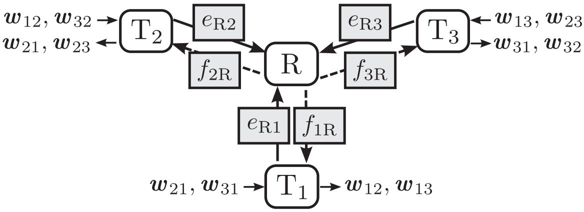


Figure 5.1: The Y-channel with three transceivers $\mathrm{T}_1, \mathrm{T}_2$ and T_3 and one intermediate relay R_1, has six independent dedicated messages $\boldsymbol{w}_{ji}$. The influence of the channel is parameterized by the corresponding $e_{\mathrm{R}i}$ and $f_{j\mathrm{R}}$. The solid arrows describe the UL and the dashed arrows the DL subchannels.

The UL-phase denotes the transmission in the first hop from the transceivers T_i to the relay R. Signals transmitted from T_i are represented by a polynomial $u_i(x)$, with messages allocated to distinct offset parameters $p_{ji}^{[t]} \in \mathbb{N}$:

$$u_i(x) \equiv \sum_{j=1, j\neq i}^{3} u_{ji}(x), \tag{5.2}$$

$$u_{ji}(x) \equiv \sum_{t=0}^{m_{ji}-1} W_{ji}^{[t]} x^{p_{ji}^{[t]}}. \tag{5.3}$$

The UL channel vector is $\boldsymbol{e} = (e_{\mathrm{R}1}, e_{\mathrm{R}2}, e_{\mathrm{R}3})$, with $e_{\mathrm{R}1}, e_{\mathrm{R}2}, e_{\mathrm{R}3} \in \mathcal{D}$. At relay R, the received signal is a shifted superposition of the three signals from all T_i:

$$r_{\mathrm{R}}(x) \equiv \boldsymbol{e}\boldsymbol{u}^{\mathsf{T}} \equiv \sum_{i=1}^{3} e_{\mathrm{R}i} u_i(x). \tag{5.4}$$

The DL-phase denotes the transmission from relay R back to the three transceivers T_j in the second hop. The DL channel is denoted by a vector $\boldsymbol{f} = (f_{1\mathrm{R}}, f_{2\mathrm{R}}, f_{3\mathrm{R}})$ with the entries $f_{1\mathrm{R}}, f_{2\mathrm{R}}, f_{3\mathrm{R}} \in \mathcal{D}$. The relay forwards its previously received superimposed signals back to all transceivers:

$$\boldsymbol{r}^{\mathsf{T}} \equiv \boldsymbol{f}^{\mathsf{T}} r_{\mathrm{R}}(x) \equiv (f_{1R}, f_{2R}, f_{3R})^{\mathsf{T}} r_{\mathrm{R}}(x). \tag{5.5}$$

The two channel vectors $\boldsymbol{e}$ and $\boldsymbol{f}$ are independent w.l.o.g. With the perfect full-duplex operation, the UL-phase and DL-phase is performed simultaneously in each time-step. But due to causality in time, a DL transmission is delayed by one time-instant w. r. t. its previous UL transmission. For distinct user-indices $i, j, k \in \mathcal{K}$ and submessage-indices t and t', the separability conditions of the 3-user Y-channel are given as follows:

Multiple-access interference conditions, for $t \in \{0, ..., m_{ji}-1\}$, $t' \in \{0, ..., m_{jk}-1\}$:

$$e_{\mathrm{R}i} f_{j\mathrm{R}} x^{p_{ji}^{[t]}} \not\equiv e_{\mathrm{R}k} f_{j\mathrm{R}} x^{p_{jk}^{[t']}} \bmod (x^n - 1) \tag{5.6}$$

$$\Rightarrow e_{\mathrm{R}i} x^{p_{ji}^{[t]}} \not\equiv e_{\mathrm{R}k} x^{p_{jk}^{[t']}} \bmod (x^n - 1), \tag{5.7}$$

intra-user interference conditions, for $t \in \{0, ..., m_{ji}-1\}$, $t' \in \{0, ..., m_{ki}-1\}$:

$$x^{p_{ji}^{[t]}} \not\equiv x^{p_{ki}^{[t']}} \bmod (x^n - 1), \tag{5.8}$$

inter-user interference conditions, for $t \in \{0, ..., m_{ji} - 1\}$, $t' \in \{0, ..., m_{ik} - 1\}$:

$$e_{\mathrm{R}i} f_{j\mathrm{R}} x^{p_{ji}^{[t]}} \not\equiv e_{\mathrm{R}k} f_{j\mathrm{R}} x^{p_{ik}^{[t']}} \mod (x^n - 1),$$
$$\Rightarrow e_{\mathrm{R}i} x^{p_{ji}^{[t]}} \not\equiv e_{\mathrm{R}k} x^{p_{ik}^{[t']}} \mod (x^n - 1) \tag{5.9}$$

Before approaching an optimal communication scheme for the Y- channel, we introduce the closely related 3-way channel.

5.1.2 Cyclic Polynomial 3-Way Channel

For the 3-way channel, the messaging matrix $\boldsymbol{M}$ is identical to (5.1). However, there are only direct links between the users and there is no intermediate relay involved. It has only one channel matrix which is defined by $\boldsymbol{D} = (d_{ji})_{1 \leq j,i \leq 3}$ with independent elements $d_{ji} \in \mathcal{D}$. The diagonal elements of $\boldsymbol{D}$ are zero. Such a 3-way channel is depicted in Figure 5.2.

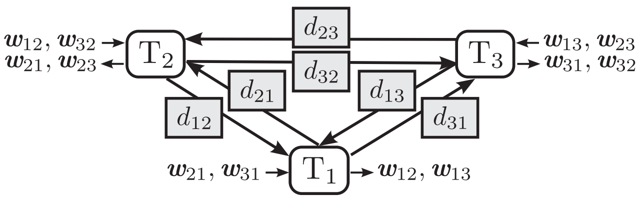


Figure 5.2: The 3-way channel with three transceivers $\mathrm{T}_1, \mathrm{T}_2$ and T_3 has six independent dedicated messages $\boldsymbol{w}_{ji}$. The influence of the channel is parameterized by the corresponding d_{ji}.

In this case, the received signal at T_j, $j \in \mathcal{K}$, is a superposition of shifted polynomials $u_i(x)$:

$$r_j(x) \equiv \textstyle\sum_{i=1}^{3} d_{ji} u_i(x) \mod (x^n - 1). \tag{5.10}$$

We can express this compactly by $\boldsymbol{r}^\top \equiv \boldsymbol{D}\boldsymbol{u}^\top \mod (x^n - 1)$, with $\boldsymbol{r} = (r_1(x), r_2(x), r_3(x))$ and $\boldsymbol{u} = (u_1(x), u_2(x), u_3(x))$.

In analogy to the 3 user Y- channel, we also define the set of separability conditions with pairwise distinct $i, j, k \in \mathcal{K}$:

Multiple-access interference conditions for $t \in \{0, ..., m_{ji} - 1\}$, $t' \in \{0, ..., m_{jl} - 1\}$:

$$d_{ji} x^{p_{ji}^{[t]}} \not\equiv d_{jl} x^{p_{jl}^{[t']}} \mod(x^n - 1), \tag{5.11}$$

intra-user interference conditions for $t \in \{1, ..., m_{ji} - 1\}$, $t' \in \{1, ..., m_{ki} - 1\}$:

$$x^{p_{ji}^{[t]}} \not\equiv x^{p_{ki}^{[t']}} \mod(x^n - 1), \tag{5.12}$$

inter-user interference conditions for $t \in \{0,...,m_{ji}-1\}$, $m' \in \{0,...,m_{ki}-1\}$:

$$d_{ji}x^{p_{ji}^{[t]}} \not\equiv d_{jl}x^{p_{ki}^{[t']}} \mod(x^n-1) \tag{5.13}$$

Having defined the channel models and the separability conditions of the Y-channel and the 3-way channel, we are now ready to investigate their upper bounds on the DoF.

5.1.3 Upper Bounds

In the case of a 3-way channel, we consider $K_\mathrm{T} = K_\mathrm{R} = 3$ and hence $\mathcal{K} = \{1,2,3\}$. At each receiver T_j, the minimal necessary number of dimensions is lower bounded by $\max(n_1,n_2,n_3)$ for $n_j = m_{ji}+m_{jk}+\max(m_{ik},m_{ki})$. The two dedicated messages received at T_j must demand at least $m_{ji}+m_{jk}$ dimensions. The intra-user interference of both transmitters T_i and T_j must cover at least $n \geq \max(m_{ik},m_{ki})$ dimensions, since each T_i must allocate its two different dedicated signals linearly independent, such that $m_{ji}+m_{ki}$ dimensions is necessary. This leads to the following expression for the upper bound on the DoF:

$$\mathrm{DoF} \leq \frac{m_{12}+m_{21}+m_{13}+m_{31}+m_{23}+m_{32}}{\max\left(n_1,n_2,n_3\right)}, \tag{5.14}$$

for distinct $i,j,k \in \mathcal{K}$. Note that this term coincides with the general upper bound on the DoF for $K_\mathrm{T} \times K_\mathrm{R}$ X-channels (cf. (3.38)) for the specific $\boldsymbol{M}$ given in (5.1):

$$\mathrm{DoF} \leq \frac{m_{12}+m_{21}+m_{13}+m_{31}+m_{23}+m_{32}}{\max\limits_{i\neq j\in\mathcal{K}}\left(\sum_{k=1,k\neq j}^{3} m_{jk}+\sum_{l=1,l\neq i}^{3} m_{li}-m_{ji}\right)}$$

For the 3-user Y-channel, the upper bound is actually the same: The transmitters T_i and T_j must satisfy the multiple-access conditions, and transmit $m_{ji}+m_{ki}$ and $m_{ij}+m_{kj}$ submessages, respectively. At relay R, the signals dedicated to be forwarded to T_k must be distinct from the intra-user interference from T_i and T_j. These dedicated signals must cover at least $m_{ki}+m_{kj}$ dimensions to be linearly decodable. The intra-user interference of T_i and T_j is aligned at R and covers at least $\max(m_{ji},m_{ij})$ dimensions. Then, relay R must forward and broadcast all received messages to all three users, since each submessage is dedicated for one receiver. This does not preclude that the received signals at T_i may be interfered with back-propagated self-interference. Hence, in order to provide a decodable signal for T_k, a number of $n_k = m_{ki}+m_{kj}+\max(m_{ji},m_{ij})$ dimensions is necessary. Combining these conditions for all users $k \in \mathcal{K}$, we obtain the lower bound:

$$n \geq \max(n_1,n_2,n_3), \tag{5.15}$$

which is obviously the same lower bound on the dimensions n as for the 3-way channel.

5.1.4 Achievable Scheme for the Y-Channel

First, we investigate an achievable scheme for the non-reciprocal 3-user Y-channel with general message lengths. A closely related scheme for a reciprocal Y-channel in the LDCM is given in [79]. Both schemes are mainly based on SA and LEaD.

Theorem 5.1. *On the* 3*-user* Y*-channel, a cyclic SA scheme achieves:*

$$\mathrm{DoF} \leq \frac{m_{12} + m_{21} + m_{13} + m_{31} + m_{23} + m_{32}}{\max(n_1, n_2, n_3)} \leq 2,$$

within $n \geq \max(n_1, n_2, n_3)$ *dimensions for distinct* $i, j, k \in \mathcal{K}$ *for:*

$$n_j = m_{ji} + m_{jk} + \max(m_{ik}, m_{ki}).$$

Proof:
(a) *Necessity of* $n \geq \max(n_1, n_2, n_3)$ *dimensions:*
The lower bound on the number of dimensions n has already been provided in Section 5.1.3.

(b) *Sufficiency of the Cyclic SA scheme:*
In the UL-phase, the signals from each T_i are aligned at R within 3 main blocks:

A) Bidirectional: There are non-zero pairs of m_{ji}, m_{ij} for two users $i \neq j$.

B) Cyclic: There are non-zero triples m_{ji}, m_{ik}, m_{kj} for three distinct users $i, j, k \in \mathcal{K}$.

C) Unidirectional: None of the above cases holds.

The communication scheme for the signal alignment at the relay R is briefly outlined as follows. For submessages satisfying (a), the bidirectional communication scheme allocates two submessages per dimension. Then, the remaining submessages satisfying (b) are allocated by the cyclic communication scheme with $\frac{3}{2}$ submessages per dimension. The yet remaining submessages are allocated by the unidirectional scheme with one submessage per dimension.

In the DL, these received signals aligned at R are simply forwarded to all T_i. The T_i use linear decoders and self-interference cancellation to resolve their dedicated signals.

Uplink

A) Bidirectional Communication

We define three parameters a_1, a_2, a_3, and their cumulative sum a, for distinct indices $i, j, k \in \mathcal{K}$:

$$a_j = \min(m_{ik}, m_{ki}), \tag{5.16}$$
$$a = a_1 + a_2 + a_3. \tag{5.17}$$

The transmitted signals of each T_i are aligned in bidirectional pairs of submessages with the following offsets of relay R:

$$\begin{array}{rlr}
\text{offset} & \text{SA at Relay R} & \text{index } t \\
x^{t}: & e_{\mathrm{R}2} x^{p_{32}^{[t]}} \equiv e_{\mathrm{R}3} x^{p_{23}^{[t]}} \bmod (x^n - 1), & t = 0, \dots, a_1 - 1, \\
x^{a_1 + t}: & e_{\mathrm{R}1} x^{p_{31}^{[t]}} \equiv e_{\mathrm{R}3} x^{p_{13}^{[t]}} \bmod (x^n - 1), & t = 0, \dots, a_2 - 1, \\
x^{a_1 + a_2 + t}: & e_{\mathrm{R}1} x^{p_{21}^{[t]}} \equiv e_{\mathrm{R}2} x^{p_{12}^{[t]}} \bmod (x^n - 1), & t = 0, \dots, a_3 - 1.
\end{array}$$

This allocation provides a unique choice for the subset the parameters $p_{ji}^{[t]}$ with the given ranges of t.

The matrix $\boldsymbol{M}'$ contains the residual submessages yet to be allocated and is denoted by:

$$\boldsymbol{M}' = \boldsymbol{M} - \boldsymbol{A}, \tag{5.18}$$

with the symmetric matrix:

$$\boldsymbol{A} = \begin{pmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix}. \tag{5.19}$$

After this allocation procedure, there are at most three non-zero elements left over in $\boldsymbol{M}'$.

B) Cyclic Communication

For two mutually exclusive cases of $\boldsymbol{M}'$, a cyclic communication scheme is utilized. We define two parameters b_1, b_2, and their cumulative sum b:

$$b_1 = \min(m'_{21}, m'_{13}, m'_{32}), \tag{5.20}$$
$$b_2 = \min(m'_{12}, m'_{23}, m'_{31}), \tag{5.21}$$
$$b = b_1 + b_2, \tag{5.22}$$

The term *cyclic* communication is due to the clock-wise/counter clock-wise cyclic indexation of the indices. Parameter $b_1 > 0$ indicates clock-wise cyclic communication, and $b_2 > 0$ indicates counter clock-wise cyclic communication. From the bidirectional communication applied before in scheme A), these two cases exclude each other:

$$\text{if } b_1 = b > 0 \Rightarrow b_2 = 0, \tag{5.23}$$
$$\text{if } b_2 = b > 0 \Rightarrow b_1 = 0, \tag{5.24}$$
$$\text{otherwise } b_1 = b_2 = 0. \tag{5.25}$$

For clock-wise cyclic communication, we align the signals at the offsets of R by:

$$\begin{array}{rlll} \text{offset} & \text{SA at Relay R} & & \text{index } t \\ x^{a+t}: & e_{\mathrm{R1}} x^{p_{21}^{[a_3+t]}} \equiv e_{\mathrm{R2}} x^{p_{32}^{[a_1+t]}} & \bmod (x^n - 1), & t = 0, ..., b_1 - 1, \\ x^{a+b_1+t}: & e_{\mathrm{R2}} x^{p_{13}^{[a_2+t]}} \equiv e_{\mathrm{R2}} x^{p_{32}^{[a_1+t]}} & \bmod (x^n - 1), & t = 0, ..., b_1 - 1. \end{array}$$

For counter clock-wise cyclic communication, we similarly align:

$$\begin{array}{rlll} \text{offset} & \text{SA at Relay R} & & \text{index } t \\ x^{a+t}: & e_{\mathrm{R2}} x^{p_{12}^{[a_3+t]}} \equiv e_{\mathrm{R1}} x^{p_{31}^{[a_2+t]}} & \bmod (x^n - 1), & t = 0, ..., b_2 - 1, \\ x^{a+b_2+t}: & e_{\mathrm{R2}} x^{p_{23}^{[a_1+t]}} \equiv e_{\mathrm{R2}} x^{p_{31}^{[a_2+t]}} & \bmod (x^n - 1), & t = 0, ..., b_2 - 1. \end{array}$$

In both of these cyclic communication schemes, the inter-user interference conditions seem to be violated. However, the receivers can linearly decode the dedicated signals in the DL using known back-propagated self-interference for cancellation. We will briefly elaborate this decoding scheme when discussing the DL-phase of this scheme. With the circulant matrix $\boldsymbol{B}$ for (either clock-wise or counter clock-wise) cyclic communication:

$$\boldsymbol{B} = \begin{pmatrix} 0 & b_2 & b_1 \\ b_1 & 0 & b_2 \\ b_2 & b_1 & 0 \end{pmatrix}, \tag{5.26}$$

the matrix $\boldsymbol{M}''$ of the residual submessages yet to be allocated is:

$$\boldsymbol{M}'' = \boldsymbol{M}' - \boldsymbol{B} = \boldsymbol{M} - \boldsymbol{A} - \boldsymbol{B}. \tag{5.27}$$

C) Unidirectional Communication

There are still at most 3 non-zero submessages in $\boldsymbol{M}''$, if scheme $b = 0$, or at most 2 non-zero submessages, if $b > 0$. Based on the previous allocation, there are six possible cases with non-zero submessages that are not yet covered by the previous schemes A and B. We define the six parameters for unidirectional communication and their cumulative sum by:

$$\begin{aligned}
&\text{Case 1}: \; c_{1,12} = m''_{12},\; c_{1,13} = m''_{13},\; c_{1,23} = m''_{23} \Rightarrow b_2 \geq 0, b_1 = 0 && (5.28)\\
&\text{Case 2}: \; c_{2,12} = m''_{12},\; c_{2,13} = m''_{13},\; c_{2,32} = m''_{32} \Rightarrow b_1 \geq 0, b_2 = 0 && (5.29)\\
&\text{Case 3}: \; c_{3,21} = m''_{21},\; c_{3,23} = m''_{23},\; c_{3,13} = m''_{13} \Rightarrow b_1 \geq 0, b_2 = 0 && (5.30)\\
&\text{Case 4}: \; c_{4,21} = m''_{21},\; c_{4,23} = m''_{23},\; c_{4,31} = m''_{31} \Rightarrow b_2 \geq 0, b_1 = 0 && (5.31)\\
&\text{Case 5}: \; c_{5,31} = m''_{31},\; c_{5,32} = m''_{32},\; c_{5,12} = m''_{12} \Rightarrow b_2 \geq 0, b_1 = 0 && (5.32)\\
&\text{Case 6}: \; c_{6,31} = m''_{31},\; c_{6,32} = m''_{32},\; c_{6,21} = m''_{21} \Rightarrow b_1 \geq 0, b_2 = 0, && (5.33)\\
&c = \textstyle\sum_{i=1}^{6} c_{i,12} + c_{i,13} + c_{i,21} + c_{i,23} + c_{i,31} + c_{i,32}. && (5.34)
\end{aligned}$$

These six cases are pairwise exclusive. The non-zero submessages in $\boldsymbol{M}''$ are each allocated to an individual frame in a multiple-access scheme. E.g., in *Case 1*, we align:

$$\begin{array}{rll}
\text{offset} & \text{R} & \text{index } t \\
x^{a+2b+t}: & e_{\mathrm{R}2} x^{p_{12}^{[a_3+b+t]}}, & t = 0, \dots, c_{1,12} - 1, \\
x^{a+2b+c_{1,12}+t}: & e_{\mathrm{R}3} x^{p_{13}^{[a_2+b+t]}}, & t = 0, \dots, c_{1,13} - 1, \\
x^{a+2b+c_{1,12}+c_{1,13}+t}: & e_{\mathrm{R}3} x^{p_{23}^{[a_1+b+t]}}, & t = 0, \dots, c_{1,23} - 1.
\end{array} \tag{5.35}$$

Note that c has at most three non-zero components in each case, e. g., $c = c_{1,12} + c_{1,13} + c_{1,23}$ in *Case 1*. The other five cases for the unidirectional communication scheme are similarly allocated at R within a frame of c dimensions. We consider all six cases in a combined view, and define the matrix $\boldsymbol{C}$ for unidirectional communication by:

$$\boldsymbol{C} = \begin{pmatrix} 0 & c_{1,12} + c_{2,12} + c_{5,12} & c_{1,13} + c_{2,13} + c_{3,13} \\ c_{6,21} + c_{4,21} + c_{3,21} & 0 & c_{4,23} + c_{1,23} + c_{3,23} \\ c_{6,31} + c_{4,31} + c_{5,31} & c_{6,32} + c_{2,32} + c_{5,32} & 0 \end{pmatrix}. \tag{5.36}$$

All submessages in the original messaging matrix $\boldsymbol{M}$ are completely allocated, if we set $\boldsymbol{C} = \boldsymbol{M}''$. The auxiliary matrices of all three schemes are combined to:

$$\boldsymbol{M} = \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}. \tag{5.37}$$

We expand n in its explicit form so that it is the maximum of all 6 lower bounds:

$$\begin{aligned} n = \max(&m_{12} + m_{13} + m_{23}, m_{12} + m_{13} + m_{32}, m_{21} + m_{23} + m_{13}, \\ &m_{21} + m_{23} + m_{13}, m_{31} + m_{32} + m_{12}, m_{31} + m_{32} + m_{21}). \end{aligned} \tag{5.38}$$

It remains to show that the separability conditions of the Y-channel hold. Since each T_i aligns its transmitted submessages to specific dimensions at R only, we do not have to take signals into account, that are aligned at other receivers. Hence, the corresponding transmit signals can be easily derived from the (aligned) received signal at $r(x)$.

As both the aligned interference signals and the dedicated signals are distinctly allocated within successively arranged frames at R, valid transmission signals always exist. The separability conditions hold by construction, if the total number of used dimensions satisfies the lower bound of n.

We now show that the proposed scheme achieves the upper bound on the DoF:

$$\begin{aligned} n &= \min(m_{23}, m_{32}) + \min(m_{13}, m_{31}) + \min(m_{12}, m_{21}) + \\ &\quad 2\min(m'_{21}, m'_{13}, m'_{32}) + 2\min(m'_{12}, m'_{31}, m'_{23}) + \\ &\quad \begin{cases} m''_{12} + m''_{13} + m''_{23}, & \text{if Case 1 holds} \\ m''_{12} + m''_{13} + m''_{32}, & \text{if Case 2 holds} \\ m''_{21} + m''_{23} + m''_{13}, & \text{if Case 3 holds} \\ m''_{21} + m''_{23} + m''_{31}, & \text{if Case 4 holds} \\ m''_{31} + m''_{32} + m''_{12}, & \text{if Case 5 holds} \\ m''_{31} + m''_{32} + m''_{21}, & \text{if Case 6 holds} \end{cases} \\ &= a_1 + a_2 + a_3 + 2b_1 + 2b_2 + \\ &\quad \begin{cases} m_{12} - a_3 - b_2 + m_{13} - a_2 - b_1 + m_{23} - a_1 - b_2, & \text{if Case 1 holds } (b_1 = 0) \\ m_{12} - a_3 - b_2 + m_{13} - a_2 - b_1 + m_{32} - a_1 - b_1, & \text{if Case 2 holds } (b_2 = 0) \\ m_{21} - a_3 - b_1 + m_{23} - a_1 - b_2 + m_{13} - a_2 - b_1, & \text{if Case 3 holds } (b_2 = 0) \\ m_{21} - a_3 - b_1 + m_{23} - a_1 - b_2 + m_{31} - a_2 - b_2, & \text{if Case 4 holds } (b_1 = 0) \\ m_{31} - a_2 - b_2 + m_{32} - a_1 - b_1 + m_{12} - a_3 - b_2, & \text{if Case 5 holds } (b_1 = 0) \\ m_{31} - a_2 - b_2 + m_{32} - a_1 - b_1 + m_{21} - a_3 - b_1, & \text{if Case 6 holds } (b_2 = 0) \end{cases} \\ &= \max(m_{12} + m_{13} + m_{23}, m_{12} + m_{13} + m_{32}, m_{21} + m_{23} + m_{13}, \\ &\qquad m_{21} + m_{23} + m_{31}, m_{31} + m_{32} + m_{12}, m_{31} + m_{32} + m_{21}). \end{aligned}$$

In other words, the number of dimensions n used for the proposed communication scheme is sufficient to convey all submessages in $\boldsymbol{M}$ for all six cases in (5.38).

Downlink

The received signal $r(x)$ of the UL at R is forwarded over the subchannels $f_{i\mathrm{R}}$ to all receivers T_i. The signals received at each T_i are only cyclically shifted versions of the forwarded signal $r(x)$ without further superimposed interference from other transmitters. Each receiver T_i cancels its back-propagated self-interference and also cancels the aforementioned interference from the cyclic communication.

For instance, T_1 cancels the known self-interference $e_{\mathrm{R}3}x^{p_{21}^{[a_3+t]}}$ and decodes the remaining non-dedicated signal $e_{\mathrm{R}2}x^{p_{32}^{[a_1+t]}}$ for $t = 0, \dots, b_1 - 1$. Next, this decoded interference signal is cancelled from the other received signal $e_{\mathrm{R}2}x^{p_{13}^{[a_2+t]}} + e_{\mathrm{R}2}x^{p_{32}^{[a_1+t]}}$, so that the dedicated signal $e_{\mathrm{R}2}x^{p_{13}^{[a_2+t]}}$ remains and can be decoded successfully. The other users perform analogous procedures for the interference in the cyclic communication schemes.

Altogether, it is shown that the given upper bound on the DoF is achievable. The DoF are only maximized if the bidirectional scheme A) can already align all dedicated messages. Hence, if $m_{23} = m_{32} = m_1, m_{13} = m_{31} = m_2$ and $m_{12} = m_{21} = m_3$ hold, we obtain $n = n_1 = n_2 = n_3 = m_1 + m_2 + m_3$, so that at most $\frac{2(m_1+m_2+m_3)}{m_1+m_2+m_3} = 2$ DoF are achievable. ■

5.1.5 Δ-Y Transformation

We have already seen that the upper bounds on the DoF of the considered 3-user Y-channel and the 3-way channel coincide exactly, even for arbitrary message lengths in $\boldsymbol{M}$. The upper bounds for the 3-user Y-channel have been shown to be achievable in the theorem above. In the following, we will show that the upper bounds of the 3-way channel are also achievable.

From the perspective of an electrical engineer, a similar equivalence occurs in wired resistor-networks to some extend. There is a so-called Δ-Y transformation[3] which is a well-known method from elementary circuit theory as, e. g., in [86], [87]. The basic statement of this transformation is that the external resistance behaviour of a Δ-shaped resistor circuit can be equivalently described by a Y-shaped resistor circuit and vice versa.

In this section, we explore a conceptually similar Δ-Y transformation for the 3-user Y-channel and the 3-way channel. To the best of our knowledge, such a transformation has not been considered in terms of multi-user interference channels in the literature yet.

To elaborate the transformation, we commence with considering a Y-channel of three users $\tilde{\mathrm{T}}_1$, $\mathrm{T}_2, \mathrm{T}_3$ and a relay R. In the first step, this channel is extended by an additional bidirectional link between two users, e. g., T_2 and T_3. The shape of a 3-way channel occurs between T_2, R and T_3. Next, we unite user $\tilde{\mathrm{T}}_1$ and relay R to a single device T_1 by permitting full cooperation for them. Now, each subchannel from T_i to T_j is renamed to d_{ji}, correspondingly. As a result, we obtain the 3-way channel.

This transformation procedure is depicted in Figure 5.3. We assume that the (virtual) subchannel between the relay R and $\tilde{\mathrm{T}}_1$ imposes no further shift w.l.o.g. The subchannels d_{23} and d_{32} represent the main difference between an ordinary Y-channel and this extended Y-channel.

[3] This transformation is also known as Π-T transform in [86], [87].

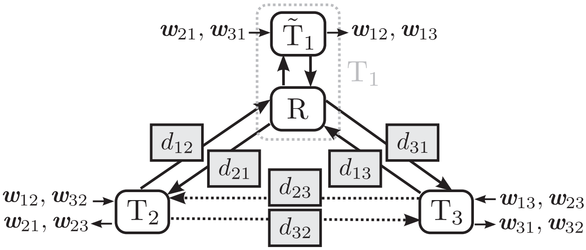


Figure 5.3: The 3 -way channel is transformed into an extended (upside-down) Y - channel including an additional bidirectional link between T_2 and T_3. The additional link is drawn with dotted lines.

5.1.6 Achievable Scheme for the 3 -Way Channel

Theorem 5.2. *Cyclic SA on the 3 -way channel achieves the upper bound:*

$$\mathrm{DoF} \leq \frac{m_{12} + m_{21} + m_{13} + m_{31} + m_{23} + m_{32}}{\max(n_1, n_2, n_3)} \leq 2,$$

within $n \geq \max(n_1, n_2, n_3)$ *for* $n_j = m_{ji} + m_{jk} + \max(m_{ik}, m_{ki})$ *with distinct* $i, j, k \in \mathcal{K}$.

Proof:
(a) *Necessity of* $n \geq \max(n_1, n_2, n_3)$ *dimensions:*
The lower bound on the number of dimensions n is already provided in Section 5.1.3.

(b) *Sufficiency of Cyclic IA for* $n = \max(n_1, n_2, n_3)$*:*
We basically apply the same achievable scheme that was used for the 3 -user Y - channel to the extended Y - channel. We only include some minor adaptations to compensate the additional interfering signals conveyed over d_{23} and d_{32}. We consider a finite communication scheme over $N+1$ time-instants with N UL-transmissions and N DL-transmissions. The entire decoding procedure at the receivers is postponed to the last, i. e., the $(N+1)$-th time-instant. The last time-instant is only a DL-transmission from R and there will be no additional interference over d_{23} and d_{32}, since the UL-transmitters of T_2 and T_3 are already silent. There are only two classes of additional interference over d_{23} and d_{32} to be analyzed:

(a) The interference over d_{ij}, for $i \neq j \in \{2, 3\}$, received at T_i is a *dedicated signal* from T_j, which will also be forwarded from R to T_i in the next time instant.

(b) The interference over d_{ij}, for $i \neq j \in \{2, 3\}$, received at T_i is an *interfering signal* from T_j that is dedicated for T_1.

Class (a): The interference of class (a) only contains messages that are dedicated for the receiver of T_2 (or of T_3). After the $(N+1)$-th time-instant, the receivers T_2 (and T_3) can completely decode all the forwarded dedicated signals that were transmitted in the UL of the last, N-th, time-instant without interference over d_{23} and d_{32}. These dedicated signals can now be used to cancel the additional interference of class (a) from the received signals of T_2 (and T_3) that were both transmitted and received during the N-th time-instant. If the interference of class (b) (this will be explained in the next paragraph) is fully removed as well, all the dedicated messages from the N-th hop can also be decoded completely. These steps of decoding and cancellation are continued in a backward decoding procedure until all interfering signals of class (a) are cancelled. Note that in the first time-instant, no DL-transmission has been performed yet. Thus, the class (a) interference is not harmful in the first time-instant and simply ignored.

Class (b): The interference of class (b) only contains messages that are dedicated for receiver T_1 and hence they represent interference for T_2 (or T_3), respectively. In this case, we apply an IN scheme. In particular, T_2 (or T_3) additionally *pre-transmits* the corresponding interference signals with a complementary sign over d_{32} (or d_{23}) of time instant l one time-instant in advance, i. e., at time-instant $l-1$. These pre-transmitted signals are aligned at the relay R, such that the forwarded signals from R in time-instant l will exactly coincide with the original interference signals of class (b) over d_{32} (or d_{23}) at time instant l. This is clearly feasible for T_2 (and T_3), since it can align its signals to any of the n levels at R. As a result, the two complementary interference signals of class (b) coincide and neutralize each other completely. Note that the pre-transmitted signals which are back-propagated to their original senders are known and can be cancelled. Moreover, since the pre-transmitted signals are dedicated signals for T_1, T_1 can cancel those signals via a backward decoding procedure as discussed analogously for the interference of class (a). The pre-transmitted signals over d_{32} (or d_{23}) arriving at T_3 (or T_2) are again neutralized by another pre-transmission. This pre-transmission procedure is iterated backwards to the first transmission, where the signals over d_{32} (or d_{23}) may be discarded at T_3 (or T_2).

Altogether, we have shown that the additional interference over the subchannels d_{32} or d_{23} can be compensated and thence we have proven that the upper bounds on the 3-way channel are achievable. ■

We conjecture that there might also be an DoF-achieving scheme for the 3-way channel that is based on IA and not on SA. Then, the subchannels d_{32} or d_{23} are also used to convey some dedicated signals directly, without relaying over T_1. Such an optimal scheme would certainly also rely on closely related bidirectional, cyclic and unidirectional schemes as discussed similarly for the 3-user Y-channel. A symmetric case of this problem is investigated in Section 5.2.7 in terms of the LDCM.

5.1.7 Discussion

In contrast to most of the previously considered channels, e. g., the 2-user X-channel, the K-user interference channel, or the $2\times2\times2$ relay-interference channel, the channel matrices of the 3-way channel and the Y-channel are not subject to further conditions, i. e., the schemes can be applied on any channel matrix $\boldsymbol{D}, \boldsymbol{E}$, and $\boldsymbol{F}$, with entries in $\mathcal{D}$. It turns out, that this is a useful property when regarding the the corresponding Gaussian MIMO 3-way channel. A direct consequence is that the problem of common eigenvectors in separate subspaces, as discussed in 3.4.2, vanishes, so that it is possible to formulate closed-form IA schemes for the MIMO 3-way channel with constant channel coefficients that are capable to achieve the upper bounds on the DoF. Note that these formulations do not resort to infinite symbol-extensions as in [3] for instance. Subsequently, we consider the 3 -way channel in terms of the LDCM and GMCM.

5.2 Capacity Region of the Reciprocal Linear Deterministic 3 -Way Channel

Similar to the CPCM, the linear deterministic 3 -way channel and the linear deterministic 3 -user Y- channel [79] are closely related. Motivated by this, we consider the reciprocal 3 -way channel in terms of the LDCM. We first provide cut-set bounds and genie-aided upper bounds describing the outer bound on the capacity region. Achieving this capacity region interestingly relies upon SA and on BD, as well. Our main tool is again a Δ-Y transformation of the 3 -way channel to an (extended) Y- channel, as motivated by elementary electrical circuit theory. However, due to the linear shift in the LDCM, the approach differs at several points. Furthermore, we integrate a capacity-achieving scheme for a symmetric 3 -way subchannel that is also based on signal-scale interference alignment (IA) as in [3], [88], and [22], without resorting to BD.

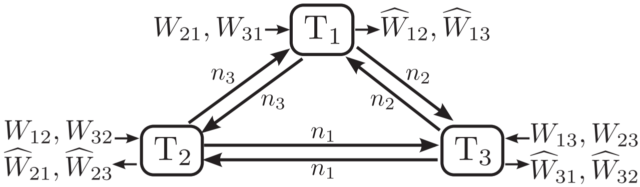


Figure 5.4: The linear deterministic 3 -way channel (or Δ- channel) with three transceivers $\mathrm{T}_1, \mathrm{T}_2$ and T_3 has six independent messages $W_{j,i}$ transmitted and six corresponding estimated messages $\widehat{W}_{j,i}$ received by the nodes, for $i \neq j \in \mathcal{K}$. The channel gains are parameterized by $n_j \in \mathbb{N}$, $j \in \mathcal{K}$.

5.2.1 System Model: Linear Deterministic 3 -Way Channel

In the linear deterministic 3 -way channel, a user T_i is a combined full-duplex transmitter Tx_i and receiver Rx_i. We consider six independent messages W_{ji} dedicated to be conveyed from T_i to T_j with $W_{ji} \in \mathcal{W}_{ji} = \{1, ..., 2^{nR_{ji}}\}$, $R_{ji} \in \mathbb{R}^+$, for distinct $i, j \in \mathcal{K} = \{1, 2, 3\}$. The vector of all messages is denoted by:

$$\boldsymbol{w} = (W_{12}, W_{21}, W_{13}, W_{31}, W_{23}, W_{32}). \tag{5.39}$$

The rate tuple $\boldsymbol{R}$ and the total sum-rate R_Σ are defined for rates $R_{ji} \in \mathbb{R}^+$ between Tx_i and Rx_j by:

$$\boldsymbol{R} = (R_{12}, R_{21}, R_{13}, R_{31}, R_{23}, R_{32}), \tag{5.40}$$
$$R_\Sigma = R_{12} + R_{21} + R_{13} + R_{31} + R_{23} + R_{32}. \tag{5.41}$$

T_j encodes its messages into a codeword[4] $\boldsymbol{x}_j^N$. The l-th symbol of $\boldsymbol{x}_j^N$ is an element of an alphabet $\mathcal{X}$ encoded as $\boldsymbol{x}_j(l) = f_{j,l}(W_{ij}, W_{kj}, \boldsymbol{y}_j^{l-1})$ for distinct $i, j, k \in \mathcal{K}$. Therein, $\boldsymbol{y}_j^{l-1}$ are all received symbols at T_j until time-instant $l-1$ with encoding function $f_{j,l}(\cdot)$. A receiving T_i decodes $(\hat{W}_{ij}, \hat{W}_{ik}) = g_i(\boldsymbol{y}_i^N, W_{ij}, W_{ik})$ with the decoding function $g_i(\cdot)$. An error occurs if $(\hat{W}_{ij}, \hat{W}_{ik}) \neq (W_{ij}, W_{ik})$. The collection of messages, encoders, and decoders defines a code for the 3-way channel. Furthermore, rate tuple $\boldsymbol{R}$ is called *achievable* if there is a sequence of codes such that the average error probability ϵ_N becomes arbitrarily small by increasing N. The set of all achievable rate tuples is the capacity region $\mathcal{C}_\Delta$.

In the LDCM, the physical channel between T_i and T_j is modelled by $n_{ji} \in \mathbb{N}$ bit pipes, and the transmitted symbols $\boldsymbol{x}_j(l)$ are bit-vectors in $\mathcal{X} = \mathbb{F}_2^q$ with a number of $q = \max_{i \neq j \in \mathcal{K}}(n_{ji})$ dimensions. The received signals $\boldsymbol{y}_i$ at receivers Rx_i, $i \in \mathcal{K}$, are deterministic functions of the transmitted signals for distinct $i, j, k \in \mathcal{K}$:

$$\boldsymbol{y}_i = \boldsymbol{S}^{q-n_{ij}} \boldsymbol{x}_j \oplus \boldsymbol{S}^{q-n_{ik}} \boldsymbol{x}_k, \tag{5.42}$$

where $\boldsymbol{S}$ is a $q \times q$ lower shift matrix, having unit entries on the lower side-diagonal. The effect of noise is mimicked by clipping linearly shifted symbols. Note that loop-back self-interference is entirely cancelled from (5.42) due to the perfect full-duplex operation.

In a wireless channel, it is valid to assume reciprocity for the bidirectional links such that we may use the following parametrization $n_{ij} = n_{ji} =: n_k$ holds for distinct $i, j, k \in \mathcal{K}$. We also assume:

$$n_3 \geq n_2 \geq n_1, \tag{5.43}$$

as an ordering of parameters w.l.o.g., and obtain $n_3 = q$. We denote this linear deterministic reciprocal 3-way channel by $\mathrm{D3C}(n_1, n_2, n_3)$.

5.2.2 Outer Bounds of Capacity Region: 3-Way Channel

Cut-set upper bounds

The cut-set bounds of broadcast and multiple-access channels as in [9] state that users T_i can not receive more bits than the number of incoming bit-levels and they cannot transmit more bits than the number of outgoing bit-levels available:

$$R_{ij} + R_{ik} \leq \max(n_k, n_j), \tag{5.44}$$

$$R_{ji} + R_{ki} \leq \max(n_k, n_j), \tag{5.45}$$

for distinct $i, j, k \in \mathcal{K}$. These cut-set bounds already provide a loose upper bound $\overline{\mathcal{C}}_{\Delta,\text{cut-set}}$ on the actual capacity region $\mathcal{C}_\Delta$. To obtain a tight capacity characterization, we include further genie-aided upper bounds similar to those derived in [79].

[4]We use $\boldsymbol{v}^N$ denote a sequence of N-vectors $(\boldsymbol{v}(1), ..., \boldsymbol{v}(N))$.

Genie-aided upper bounds

Receiver T_1 intends to decode the dedicated messages W_{12} and W_{13} using its received signal $\boldsymbol{y}_1^N$ and its own messages W_{21}, W_{31}, with a reliable decoding strategy. Let the interfering message W_{23} be provided to node T_1 as genie-aided side-information. Since T_1 already knows W_{13} and W_{23}, it can reconstruct $\boldsymbol{x}_3(1)$. From $\boldsymbol{y}_1^N$ and $\boldsymbol{x}_3(1)$, T_1 can derive $\boldsymbol{x}_2(1)$ from the deterministic function (5.42). T_1 then constructs $\boldsymbol{y}_3(1)$ from $\boldsymbol{x}_1(1)$ and $\boldsymbol{x}_2(1)$. With W_{13}, W_{23} and $\boldsymbol{y}_3(1)$, T_1 can generate $\boldsymbol{x}_3(2)$. These steps are repeated accordingly for all time-instants from 2 to N until $\boldsymbol{y}_3^N$ is completely constructed. Therefore, by knowing $\boldsymbol{y}_1^N, W_{21}, W_{31}$ and W_{23} at T_1, it can reliably decode W_{12} and W_{13}, and then reconstruct $\boldsymbol{y}_3^N$ to reliably decode W_{32}. All messages are known at node T_1 now. Thus, for the genie-aided channel, any reliable code allows decoding W_{32}. From Fano's inequality [45], we can derive:

$$
\begin{aligned}
& N(R_{12}+R_{13}+R_{32}) && (5.46)\\
&\le I(W_{12}, W_{13}, W_{32}; \boldsymbol{y}_1^N, W_{21}, W_{31}, W_{23}) + N\epsilon_N \\
&\le \mathrm{H}(\boldsymbol{y}_1^N) - \mathrm{H}(\boldsymbol{y}_1^N|\boldsymbol{w}) + N\epsilon_N \\
&\le \mathrm{H}(\boldsymbol{y}_1^N) + N\epsilon_N \\
&\le N(\max(n_3, n_2) + \epsilon_N) \\
&= N(n_3 + \epsilon_N),
\end{aligned}
$$

where $\epsilon_N \to 0$ as $N \to \infty$. By letting $N \to \infty$, we get $R_{12}+R_{13}+R_{32} \le n_3$. Similar bounds can be derived by considering different receivers and side-information (cf. Appendix C).

Outer Bound Region

By considering the genie-aided and non-redundant cut-set bounds jointly, we obtain the following set of upper bounds on the capacity region $\mathcal{C}_\Delta$:

$$R_{31} + R_{32} \le n_2, \tag{5.47}$$
$$R_{13} + R_{23} \le n_2, \tag{5.48}$$
$$R_{12} + R_{13} + R_{32} \le n_3, \tag{5.49}$$
$$R_{12} + R_{13} + R_{23} \le n_3, \tag{5.50}$$
$$R_{21} + R_{23} + R_{13} \le n_3 + n_2 - n_1, \tag{5.51}$$
$$R_{21} + R_{23} + R_{31} \le n_3, \tag{5.52}$$
$$R_{31} + R_{32} + R_{21} \le n_3, \tag{5.53}$$
$$R_{31} + R_{32} + R_{12} \le n_3 + n_2 - n_1. \tag{5.54}$$

From (5.49) and (5.52) for instance, the sum-capacity upper bound yields $R_\Sigma \le 2n_3$. This set of bounds leads to the following lemma. Note that this sum-rate upper bound coincides with the sum-rate upper bound of the (reciprocal) deterministic two-way channel [89].

Lemma 5.3. *The capacity region $\mathcal{C}_\Delta$ of the* D3C(n_1, n_2, n_3) *is outer bounded by $\overline{\mathcal{C}}_\Delta$, i.e., $\mathcal{C}_\Delta \subseteq \overline{\mathcal{C}}_\Delta$, where:*

$$\overline{\mathcal{C}}_\Delta = \{\boldsymbol{R} \in \mathbb{R}_+^6 \mid \boldsymbol{R} \textit{ satisfies } (5.47)\text{-}(5.54)\}.$$

This outer bound is in fact achievable. The achievability of this bound is proven via a Δ-Y transformation utilizing the optimal scheme for the Y-channel as a building block. Next, we briefly recapitulate the related Y-channel and its capacity region in terms of the LDCM as characterized in [79].

5.2.3 System Model: Linear Deterministic Y-Channel

The linear deterministic reciprocal 3-user Y-channel[5] DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ is depicted in Figure 5.5. The definitions of the message vector, the rate tuple, the transmission sym-

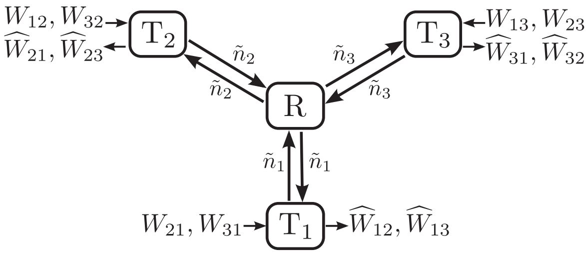


Figure 5.5: The reciprocal Y-channel with three transceivers $\mathrm{T}_1, \mathrm{T}_2$, and T_3 has six independent messages W_{ji} transmitted and six corresponding estimated messages $\widehat{W}_{ji}$ received by the nodes, $i \neq j \in \mathcal{K}$. The channel gains are parameterized by $\tilde{n}_j \in \mathbb{N}$, for $j \in \mathcal{K}$.

bols and the encoding/decoding functions carry over from those given in Section 5.2.1, but are denoted with the tilde-notation to distinguish between the two models. In contrast to the 3-way channel, all users T_j are connected via bidirectional reciprocal links to an intermediate relay R. The channel gain from R to user T_j is denoted by $\tilde{n}_j$. The gains are ordered w.l.o.g. by:

$$\tilde{n}_1 \geq \tilde{n}_2 \geq \tilde{n}_3, \tag{5.55}$$

so that $q = \max_{i \in \mathcal{K}}(\tilde{n}_i) = \tilde{n}_1$. Note that this ordering is reversely oriented when comparing it with (5.43). The transmitted signals are vectors $\boldsymbol{x}_j, \boldsymbol{x}_\mathrm{R} \in \mathbb{F}_2^q$ from T_j and R, respectively. The received signal at R and the received signals at T_j are given by

$$\boldsymbol{y}_\mathrm{R} = \textstyle\sum_{j=1}^{3} \boldsymbol{S}^{q-n_j} \boldsymbol{x}_j, \tag{5.56}$$

$$\boldsymbol{y}_j = \boldsymbol{S}^{q-n_j} \boldsymbol{x}_\mathrm{R}, \tag{5.57}$$

respectively. Next, we re-state the capacity region of the linear shift deterministic Y-channel, which will be an essential part of the proof of the achievability of Lemma 5.3.

[5]Our notation slightly differs from [79] w.r.t. swapped indexation and tilde.

5.2.4 Outer Bounds of Capacity Region: Y- Channel

The capacity region $\mathcal{C}_Y$ of the DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ has already been characterized in [79], and is given by the set of rate tuples $\tilde{\boldsymbol{R}} = (\tilde{R}_{12}, \tilde{R}_{21}, \tilde{R}_{13}, \tilde{R}_{31}, \tilde{R}_{23}, \tilde{R}_{32})$, satisfying:

$$\tilde{R}_{31} + \tilde{R}_{32} \le \tilde{n}_3, \tag{5.58}$$
$$\tilde{R}_{13} + \tilde{R}_{23} \le \tilde{n}_3, \tag{5.59}$$
$$\tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_{32} \le \tilde{n}_2, \tag{5.60}$$
$$\tilde{R}_{12} + \tilde{R}_{13} + \tilde{R}_{23} \le \tilde{n}_2, \tag{5.61}$$
$$\tilde{R}_{21} + \tilde{R}_{23} + \tilde{R}_{13} \le \tilde{n}_1, \tag{5.62}$$
$$\tilde{R}_{21} + \tilde{R}_{23} + \tilde{R}_{31} \le \tilde{n}_2, \tag{5.63}$$
$$\tilde{R}_{31} + \tilde{R}_{32} + \tilde{R}_{21} \le \tilde{n}_2, \tag{5.64}$$
$$\tilde{R}_{31} + \tilde{R}_{32} + \tilde{R}_{12} \le \tilde{n}_1. \tag{5.65}$$

There is an interesting resemblance between the bounds (5.58)-(5.65) and (5.47)-(5.54). This resemblance will be exploited to design an optimal scheme for the 3-way channel next.

5.2.5 Δ-Y Transformation

Equating the upper bounds of the D3C(n_1, n_2, n_3) in (5.47)-(5.54) and the upper bounds of DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$ in (5.58)-(5.65) yields:

$$\tilde{n}_1 = n_2 + n_3 - n_1 \tag{5.66}$$
$$\tilde{n}_2 = n_3, \tag{5.67}$$
$$\tilde{n}_3 = n_2. \tag{5.68}$$

In other words, the outer bound for the D3C(n_1, n_2, n_3) coincides with the capacity region of a DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$. Note that the ordering of the channel gains in (5.43) and (5.55) still holds.

In order to show the achievability of the outer bound in Lemma 5.3, we first express the 3-way channel in terms of an extended Y- channel as depicted in Figure 5.6 (cf. Section 5.1.5 for the CPCM). User T_1 is extended such that it behaves like a virtual relay R which is also connected to a virtual user $\tilde{\mathrm{T}}_1$ via an artificial sub-channel parametrized by $\tilde{n}_1$. At R, the topmost levels $\tilde{n}_2 + 1, ..., \tilde{n}_1$ are only accessible by $\tilde{\mathrm{T}}_1$ and not visible for T_2 and T_3 (see Figure 5.7) and hence only virtual within T_1. The residual link with n_1 from the previous D3C(n_1, n_2, n_3) remains as a weak bidirectional link between T_2 and T_3 in the extended Y- channel eDYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, n_1)$. The channel gain n_1 is still the weakest one, since:

$$n_1 \le n_2 = \tilde{n}_3 \le \tilde{n}_2 = n_3 \le n_3 + n_2 - n_1 = \tilde{n}_1. \tag{5.69}$$

The optimal scheme for the Y- channel already achieves the outer bound $\overline{\mathcal{C}}_\Delta$ for $n_1 = 0$. However, in general we have $n_1 \ge 0$. Hence, we have to modify our scheme to deal with the additional interference signals over n_1.

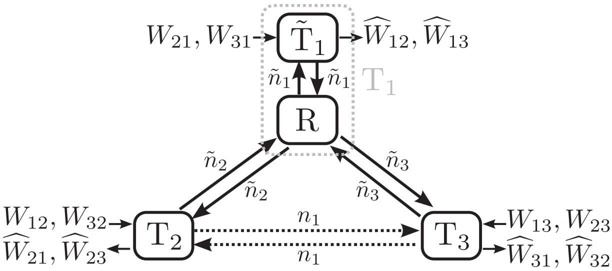


Figure 5.6: The reciprocal 3-way channel D3C(n_1, n_2, n_3) is transformed into an extended (upside-down) Y-channel eDYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, n_1)$ including an additional bidirectional link n_1 between T_2 and T_3.

5.2.6 Achievability of $\overline{\mathcal{C}}_\Delta$

Let all (virtual) users $\tilde{\mathrm{T}}_1$, T_2, T_3 and R apply the capacity-achieving SA scheme described in [79] as if it would be applied on a DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, but without decoding at the receivers yet. We consider N uplink and N transmissions, over $N+1$ time-instants. We call this scheme the 'original' scheme. In Figures 5.7 and 5.8, we depict a signal-level representation of the eDYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, n_1)$ with the uplink on top, the downlink at the bottom, and with the additional interference links with channel gain n_1 crossing in the middle.

In contrast to the original scheme for the DYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, the achievable scheme for the eDYC$(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, n_1)$ must be adapted to deal with the signals inherently transmitted over n_1. We will overlay the adapted scheme on top of the original scheme.

In particular, any signal transmitted by T_i, $i \in \{2, 3\}$, appearing on the topmost levels $\tilde{n}_1 - n_1 + 1, ..., \tilde{n}_1$, will interfere at receiver T_j on the lowermost levels $1, ..., n_1$. We discern three classes of interference over n_1 that are potentially received at T_2 and T_3 when applying the original scheme:

(a) The interference over n_1 received at T_i is a dedicated signal from T_j to T_i, which will also be forwarded from R to T_i in the next time-instant.

(b) The interference over n_1 received at T_3 is a dedicated signal from T_2 to T_1.

(c) The interference over n_1 received at T_2 is a dedicated signal from T_3 to T_1.

The signal levels of the uplink to R for instance comprise three components: the topmost $\{\tilde{n}_2+1, ..., \tilde{n}_1\}$ levels accessible by T_1 only, the levels $\{\tilde{n}_3+1, ..., \tilde{n}_2\}$ accessible by T_1 and T_2, and the lowermost $\{1, ..., \tilde{n}_3\}$ levels accessible by all three uplink users.

Class (a): To compensate interference of class (a), we postpone decoding until the last signals of the $(N+1)$-th time-instant are received. The transmission scheme does not change w.r.t. the original one. Since the (uplink) transmitters of T_2 and T_3 are silent on the $(N+1)$-th time-instant, the (downlink) receivers of T_2 and T_3 receive no signal over n_1 at the $(N+1)$-th time-instant. Hence, T_2 and T_3 can decode their dedicated signals as received in the last hop. In fact, the downlink signals of the $(N+1)$-th time-instant behave analogously to the $(N+1)$-th hop of the original scheme. Since the class (a) interference of the N-th hop is a subset of the dedicated signals in the $(N+1)$-th hop, it is cancelled after decoding the dedicated signals of the $(N+1)$-th hop. With such a BD scheme, the interference of class (a) is cancelled analogously for all preceding time-instants $N-1, ..., 2, 1$.

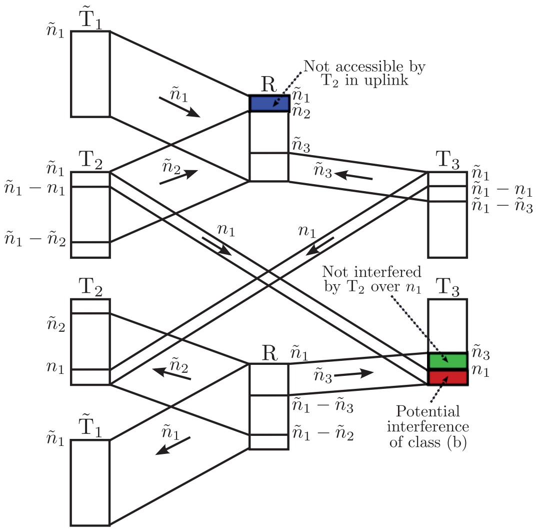


Figure 5.7: Auxiliary illustration for interference of class (b).

Class (b): To compensate the interference of class (b), i. e., those bits received at T_3 over n_1 carrying a dedicated signal from T_2 to T_1, say $\boldsymbol{x}_{12}$, we apply an IN scheme. In detail, T_2 *pre-transmits* the interference signal $(\boldsymbol{x}_{12}(l))$ one time-instant in advance (in time-instant $l-1$) as follows. Assume that T_3 receives $[\boldsymbol{x}'^{\mathsf{T}}_{R,3}(l), \boldsymbol{x}^{\mathsf{T}}_{R,3}(l)]^{\mathsf{T}}$ from R in the downlink at time-instant l, where $\boldsymbol{x}'_{R,3}(l)$ and $\boldsymbol{x}_{R,3}(l)$ are binary vectors of lengths n_1 and $\tilde{n}_3 - n_1$, respectively (see Figure 5.7). Moreover, assume that T_3 receives interference from $\boldsymbol{x}_{12}(l)$ over some bits of $\boldsymbol{x}_{R,3}(l)$. To deal with this interference, T_2 pre-transmits $\boldsymbol{x}_{12}(l)$ in time-instant $l-1$ in the uplink, over exactly the same levels where $\boldsymbol{x}_{R,3}(l)$ is received in the uplink[6]. By doing so, T_3 receives $\boldsymbol{x}_{12}(l)$ twice over $\boldsymbol{x}_{R,3}(l)$ in the downlink, once from T_2 and once from R. Since $\boldsymbol{x}_{12}(l)$ is a binary vector, the addition of $\boldsymbol{x}_{12}(l)$ to itself results in interference neutralization.

It remains to make sure that the pre-transmission does not disturb any other node. Clearly, $\boldsymbol{x}_{12}$ does not disturb T_2 since it originates from the same node T_2. Also, $\boldsymbol{x}_{12}$ does not disturb T_1 since $\boldsymbol{x}_{12}$ is a desired signal at T_1, and thus the interfering $\boldsymbol{x}_{12}$ is removed by BD.

One more problem remains. Our approach using IN only works if $\boldsymbol{x}_{R,3}(l)$ is received over levels that are accessible by T_2 in the uplink, i. e., the levels $1, ..., \tilde{n}_2$ at R. However,

[6]Note that $\boldsymbol{x}_{R,3}(l)$ is received at R at time-instant $l-1$, and transmitted at time-instant l.

$\boldsymbol{x}_{R,3}(l)$ might contain information from T_1, say $\boldsymbol{x}_{31}$, which might not be accessible by T_2 in the uplink. This is exactly the case if T_1 sends $\boldsymbol{x}_{31}$ over levels $\tilde{n}_2 + 1, ..., \tilde{n}_1$ at R (blue area in Figure 5.7). However, the given problem can be solved easily by noting that the number of levels in the blue area in Figure 5.7 is $\tilde{n}_1 - \tilde{n}_2$. We have $\tilde{n}_1 - \tilde{n}_2 = \tilde{n}_3 - n_1$ by (5.66), i. e., the same number of levels in the non-interfered downlink levels at T_3 (green area in Figure 5.7). Therefore, we exploit this interesting equality of the transformation: R forwards $\boldsymbol{x}_{31}$ over the non-interfered downlink levels at T_3 and the given problem is avoided. By pursuing such an approach, the impact of class (b) interference is completely eliminated.

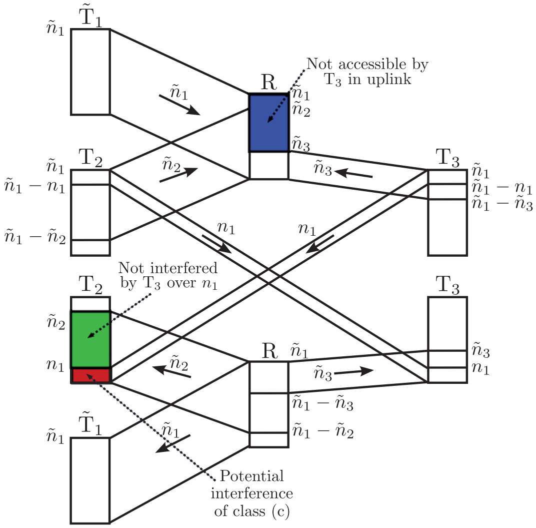


Figure 5.8: Auxiliary illustration for interference of class (c).

Class (c): To compensate the interference of class (c) at T_2 received over n_1, i. e., a dedicated signal $\boldsymbol{x}_{31}$ from T_3 to T_1, we apply a similar IN scheme. T_3 likewise additionally pre-transmits class (c) interference one time-instant in advance. T_3 can access $\tilde{n}_3$ levels in the uplink to R that are potentially forwarded to T_2 in the downlink during the next time-instant.

However, the levels $\tilde{n}_3 + 1, ..., \tilde{n}_1$ (blue area in Figure 5.8) are not accessible by T_3 in the uplink. Thus, if the signal received by T_2 from R in the downlink over levels $1, ..., n_1$ are sent over relay levels $\tilde{n}_3 + 1, ..., \tilde{n}_1$ in the uplink, then, T_3 can not perform IN. However this scenario can be avoided by sending all signals received in the uplink on the blue levels in Figure 5.8, over the green levels in the downlink. This is possible since $\tilde{n}_2 - n_1 = \tilde{n}_1 - \tilde{n}_3$ holds by (5.66). In this case, these signals do not interfere with the levels $1, ..., n_1$ at T_2 which renders T_3 capable of performing IN.

For the downlink from R to T_3, the pre-transmitted signals which are back-propagated to T_3 are known self-interference and cancelled. For the downlink of T_1, these pre-transmitted signals are dedicated for T_1 and cancelled by BD. Thus, the interference of class (c) is eliminated as well.

In all three classes, the interference over the bidirectional link n_1 is cancelled or neutralized and all dedicated signals are decodable by BD after $N + 1$ time-instants. Altogether, this proves the achievability of $\overline{\mathcal{C}}_\Delta$ leading to the following Theorem.

Theorem 5.4. *The capacity region $\mathcal{C}_\Delta$ of the* D3C(n_1, n_2, n_3) *is given by $\overline{\mathcal{C}}_\Delta$ defined in Lemma 5.5.*

At this point, the achievability of the capacity region is fully proven by means of SA, IN and BD.

5.2.7 Capacity Region of the Symmetric Case by IA

Interestingly, signals conveyed over the weak link n_1 are not used for direct communication. The interfering signals over n_1 are cancelled or neutralized by the communication scheme proposed in Section 5.2.6, so that the impact of the link n_1 is effectively eliminated. A certain drawback of our previous scheme is that the receivers must wait for $N + 1$ time-instants to apply the BD procedure. This is a very limiting property, especially for delay-limited communications [90].

As a contrary approach, we now propose a purely IA-based communication scheme for the symmetric D3C(m, m, m) that achieves the corresponding capacity region. The communication scheme for the D3C(m, m, m) based on IA is proven with similar methods as the one in [79]. In this case, there is no need for BD and IN.

Theorem 5.5. *An interference alignment scheme based on bidirectional, cyclic and unidirectional communication suffices to achieve the outer bounds on the capacity region of a symmetric* D3C(m, m, m) *with $m \in \mathbb{N}$.*

Proof:
We consider a communication scheme of three components (cf. Theorem 5.1):

A) Bidirectional: For distinct $i, j \in \mathcal{K}$, the pair of rates R_{ji}, R_{ij} is non-zero.

B) Cyclic: For distinct $i, j, k \in \mathcal{K}$, the triple of rates R_{ji}, R_{jk}, R_{ki} is non-zero, whereas $R_{ij} = R_{kj} = R_{ik} = 0$.

C) Unidirectional: None of the above cases holds.

We now outline our proposed IA scheme based on the components A, B and C. We will begin with scheme A on the D3C(m, m, m) operating at 2 bits per level. Pairs of users communicate bidirectionally. Then, we reduce the channel to D3C(m', m', m') by removing the already used levels from scheme A. Next, scheme B with $3/2$ bits per level is applied. Again we reduce the channel to D3C(m'', m'', m'') removing the levels occupied by scheme B. In the last step, we apply scheme C allocating 1 bit per level. If the rate tuple to be achieved does not satisfy one of conditions A, B, and C, the corresponding scheme is merely discarded. We will show in the following that these schemes suffice to achieve the outer bounds of the capacity region for the D3C(m, m, m).

A) Bidirectional Communication on the $\mathrm{D3C}(m,m,m)$

We define the following transmission parameters $a,b,c \in \mathbb{N}$:

$$a = \min(R_{12}, R_{21}),\; b = \min(R_{13}, R_{31}),\; c = \min(R_{23}, R_{32}). \tag{5.70}$$

If $a = b = c = 0$ holds, scheme A is skipped and we continue with scheme B. We propose a signal allocation such that 2 bits per level are achieved. The signals are $\boldsymbol{x}_{12}, \boldsymbol{x}_{21} \in \mathbb{F}_2^a$, $\boldsymbol{x}_{31}, \boldsymbol{x}_{13} \in \mathbb{F}_2^b$, and $\boldsymbol{x}_{32}, \boldsymbol{x}_{23} \in \mathbb{F}_2^c$. To transmit these signals, $a+b+c \le m$ levels are allocated as depicted in Figure 5.9. The interference signals $\boldsymbol{x}_{ji}$ and $\boldsymbol{x}_{ij}$ are aligned at T_k with pairwise distinct $i,j,k \in \mathcal{K}$.

k	$\boldsymbol{x}_k$	$\boldsymbol{y}_k$	intervals of levels
1	0	$\boldsymbol{x}_{32}+\boldsymbol{x}_{23}$	$a+b+1,\dots,a+b+c$
1	$\boldsymbol{x}_{31}$	$\boldsymbol{x}_{13}$	$a+1,\dots,a+b$
1	$\boldsymbol{x}_{21}$	$\boldsymbol{x}_{12}$	$1,\dots,a$
2	$\boldsymbol{x}_{32}$	$\boldsymbol{x}_{23}$	$a+b+1,\dots,a+b+c$
2	0	$\boldsymbol{x}_{13}+\boldsymbol{x}_{31}$	$a+1,\dots,a+b$
2	$\boldsymbol{x}_{12}$	$\boldsymbol{x}_{21}$	$1,\dots,a$
3	$\boldsymbol{x}_{23}$	$\boldsymbol{x}_{32}$	$a+b+1,\dots,a+b+c$
3	$\boldsymbol{x}_{13}$	$\boldsymbol{x}_{31}$	$a+1,\dots,a+b$
3	0	$\boldsymbol{x}_{21}+\boldsymbol{x}_{12}$	$1,\dots,a$

Figure 5.9: A) Allocation of signals to bit-levels for the bidirectional scheme over the $\mathrm{D3C}(m,m,m)$. The 1st column denotes the considered user, the 2nd column the transmitted signal, the 3rd column its received signal, and the 4th column describes the interval of levels concerned. The lowest bit-level is indexed by 1.

This allocation scheme is only feasible if enough levels are available at the transmitters and receivers for all bidirectional streams. For $\boldsymbol{R} \in \overline{\mathcal{C}}$, the following must hold on a,b,c:

$$a+b+c \overset{(5.70)}{\le} R_{12}+R_{13}+R_{32} \overset{(5.49)}{\le} m. \tag{5.71}$$

This is also true for all other upper bounds. For the yet unused levels, we still need to achieve the residual rate-vector:

$$\begin{aligned}\boldsymbol{R}' &= (R_{12}-a, R_{21}-a, R_{13}-b, R_{31}-b, R_{23}-c, R_{32}-c)\\ &= (R'_{12}, R'_{21}, R'_{13}, R'_{31}, R'_{23}, R'_{32}).\end{aligned} \tag{5.72}$$

So far, at least three components will already be zero due to the min-expressions in (5.70). We remove the allocated levels so that the reduced $\mathrm{D3C}(m',m',m')$ is parameterized by:

$$m' = m-a-b-c. \tag{5.73}$$

Clearly, the reduced channel remains symmetric.

i	$\boldsymbol{x}_i$	$\boldsymbol{y}_i$	intervals of levels
1	0	$\boldsymbol{x}_{13}+\boldsymbol{x}_{32}$	$d+1, ..., 2d$
1	$\boldsymbol{x}_{21}$	$\boldsymbol{x}_{32}$	$1, ..., d$
2	$\boldsymbol{x}_{32}$	$\boldsymbol{x}_{13}$	$d+1, ..., 2d$
2	$\boldsymbol{x}_{32}$	$\boldsymbol{x}_{21}$	$1, ..., d$
3	$\boldsymbol{x}_{13}$	$\boldsymbol{x}_{32}$	$d+1, ..., 2d$
3	0	$\boldsymbol{x}_{21}+\boldsymbol{x}_{32}$	$1, ..., d$

Figure 5.10: B) Allocation of signals to bit-levels for D3C(m', m', m') for clock-wise cyclic communication.

i	$\boldsymbol{x}_i$	$\boldsymbol{y}_i$	intervals of levels
1	0	$\boldsymbol{x}_{12}+\boldsymbol{x}_{23}$	$e+1, ..., 2e$
1	$\boldsymbol{x}_{31}$	$\boldsymbol{x}_{12}$	$1, ..., e$
2	$\boldsymbol{x}_{12}$	$\boldsymbol{x}_{23}$	$e+1, ..., 2e$
2	$\boldsymbol{x}_{12}$	$\boldsymbol{x}_{23}$	$1, ..., e$
3	$\boldsymbol{x}_{23}$	$\boldsymbol{x}_{12}$	$e+1, ..., 2e$
3	0	$\boldsymbol{x}_{12}+\boldsymbol{x}_{31}$	$1, ..., e$

Figure 5.11: B) Allocation of signals to bit-levels for D3C(m', m', m') for counter clock-wise cyclic communication.

B) Cyclic Communication on D3C(m', m', m')

Given that the conditions for scheme B hold, and depending on the residual rate-vector $\boldsymbol{R}'$ computed in (5.72), we apply either clock-wise cyclic communication $1 \to 2 \to 3 \to 1$ with parameter d or counter-clock-wise cyclic communication $1 \to 3 \to 2 \to 1$ with parameter e. The parameters $d, e \in \mathbb{N}$ are:

$$d = \min(R'_{21}, R'_{13}, R'_{32}), \ \ e = \min(R'_{12}, R'_{31}, R'_{23}). \tag{5.74}$$

Note that either d or e must be zero, since bidirectional communication is already taken care of by the previous scheme A. The definitions in (5.74) provide two cases:

$$d > 0 \Rightarrow e = 0, \ \ a = R_{12}, \ \ b = R_{31}, \ \ c = R_{23}, \tag{5.75}$$
$$e > 0 \Rightarrow d = 0, \ \ a = R_{21}, \ \ b = R_{13}, \ \ c = R_{32}. \tag{5.76}$$

If both $d = e = 0$, this section is skipped and we continue with scheme C. In the following, we propose a signal allocation scheme such that $\frac{3}{2}$ bits per level are achieved.

For case (5.75), $R'_{12} = R'_{31} = R'_{23} = 0$ and $R'_{21}, R'_{13}, R'_{32}$ are non-zero. The signals are $\boldsymbol{x}_{21}$, $\boldsymbol{x}_{13}$, $\boldsymbol{x}_{32} \in \mathbb{F}_2^d$, allocated in blocks of d levels as depicted in Figure 5.10. The constraint $2d \leq m'$ must hold at each user to provide a feasible allocation. Signal $\boldsymbol{x}_{32}$ is transmitted by T_2 on both intervals of size d and T_1 applies interference cancellation to decode $\boldsymbol{x}_{13}$ from $\boldsymbol{x}_{13} + \boldsymbol{x}_{32}$. The interference signals $\boldsymbol{x}_{21}$ and $\boldsymbol{x}_{32}$ are aligned at T_3.

Sufficiently many levels are available for scheme B on the reduced D3C(m', m', m')

if $\boldsymbol{R} \in \overline{\mathcal{C}}$, since:

$$\begin{aligned} 2d &\overset{(5.74)}{\leq} R'_{13} + R'_{32} \\ &\overset{(5.72)}{=} R_{13} + R_{32} - b - c \\ &\overset{(5.47)}{\leq} m - R_{12} - b - c \\ &\overset{(5.75)}{=} m - a - b - c \overset{(5.73)}{=} m'. \end{aligned} \tag{5.77}$$

The second case (5.76) for counter clock-wise communication is derived analogously, but with the indices swapped and with an adapted allocation (cf. Figure 5.11). In particular, we have $R'_{21} = R'_{13} = R'_{32} = 0$ and $R'_{12}, R'_{31}, R'_{23}$ are non-zero. In analogy to (5.77), the allocations for counter-clock-wise communication with parameter e satisfies $\boldsymbol{R} \in \overline{\mathcal{C}}$:

$$2e \overset{(5.74)}{\leq} R'_{31} + R'_{23} \leq m - a - b - c = m'.$$

For the yet unused levels, the residual rate-vector is:

$$\begin{aligned} \boldsymbol{R}'' &= (R'_{12} - d, R'_{21} - e, R'_{13} - e, R'_{31} - d, R'_{23} - d, R'_{32} - e) \\ &= (R''_{12}, R''_{21}, R''_{13}, R''_{31}, R''_{23}, R''_{32}), \end{aligned} \tag{5.78}$$

over the $\mathrm{D3C}(m'', m'', m'')$ (where either d or e is zero) with:

$$m'' = m' - 2d - 2e. \tag{5.79}$$

C) Unidirectional Communication on the $\mathrm{D3C}(m'', m'', m'')$

Six possible non-zero rate tuples remain that are not yet covered by the previous schemes A and B:

$$\begin{aligned} &(R''_{21}, R''_{31}, R''_{32}) \neq \mathbf{0}_{1\times 3},\ (R''_{21}, R''_{31}, R''_{23}) \neq \mathbf{0}_{1\times 3}, \\ &(R''_{12}, R''_{13}, R''_{23}) \neq \mathbf{0}_{1\times 3},\ (R''_{12}, R''_{13}, R''_{32}) \neq \mathbf{0}_{1\times 3}, \\ &(R''_{12}, R''_{31}, R''_{32}) \neq \mathbf{0}_{1\times 3},\ (R''_{21}, R''_{13}, R''_{23}) \neq \mathbf{0}_{1\times 3}. \end{aligned}$$

These cases pairwise exclude each other. W.l.o.g., we only consider the unidirectional case $(R''_{12}, R''_{13}, R''_{23}) \neq \mathbf{0}_{1\times 3}$, here. The remaining cases are derived by analogous steps. We have $R''_{21} = R''_{31} = R''_{32} = 0$ and we parameterize the 3 non-zero rates by $f, g, h \in \mathbb{N}$:

$$R''_{12} = f,\ R''_{13} = g,\ R''_{23} = h. \tag{5.80}$$

The signals are $\boldsymbol{x}_{12} \in \mathbb{F}_2^f$, $\boldsymbol{x}_{13} \in \mathbb{F}_2^g$, $\boldsymbol{x}_{23} \in \mathbb{F}_2^h$. A number of $f + g + h \leq m''$ levels are allocated as depicted in Figure 5.12.

Since we demand $\boldsymbol{R}'' \in \overline{\mathcal{C}}$, we discern two cases depending on the previous scheme B. In the first case, we assume that clock-wise cyclic communication was applied before.

k	$\boldsymbol{x}_k$	$\boldsymbol{y}_k$	intervals of levels
3	0	$\boldsymbol{x}_{23}$	$f+g+1,\ldots,f+g+h$
3	0	$\boldsymbol{x}_{13}$	$f+1,\ldots,f+g$
3	0	$\boldsymbol{x}_{12}$	$1,\ldots,f$
1	0	$\boldsymbol{x}_{23}$	$f+g+1,\ldots,f+g+h$
1	0	$\boldsymbol{x}_{13}$	$f+1,\ldots,f+g$
1	$\boldsymbol{x}_{12}$	0	$1,\ldots,f$
2	$\boldsymbol{x}_{23}$	$\boldsymbol{x}_{12}$	$f+g+1,\ldots,f+g+h$
2	$\boldsymbol{x}_{13}$	0	$f+1,\ldots,f+g$
2	0	0	$1,\ldots,f$

Figure 5.12: C) Allocation of signals to bit-levels in unidirectional communication over D3C(m'',m'',m'').

Recall that either d or e must be zero. If $d>0$ and $e=0$, then:

$$\begin{aligned} f+g+h &= R''_{12}+R''_{13}+R''_{23} \\ &\overset{(5.78)}{=} R'_{12}+R'_{13}+R'_{23}-2d \\ &\overset{(5.72)}{=} R_{12}+R_{13}+R_{23}-a-b-c-2d \\ &\overset{(5.47)}{\leq} m-a-b-c-2d \\ &\overset{(5.73)}{=} m'-2d. \end{aligned} \tag{5.81}$$

Otherwise, if $e>0$ (counter clock-wise) and $d=0$, then:

$$\begin{aligned} f+h+g &\overset{(5.78),(5.72)}{=} R_{12}+R_{13}+R_{23}-a-b-c-e \\ &\overset{(5.76)}{\leq} R_{12}+R_{13}-a-b-e \\ &\overset{(5.52)}{\leq} m-R_{23}-a-b-e \\ &\leq m-(e+c)-a-b-e && (5.82) \\ &= m'-2e. && (5.83) \end{aligned}$$

For (5.82), we use $e \leq R'_{23} = R_{23}-c$ from (5.80) and (5.72). Since either d or e is zero, combining (5.81) and (5.83) yields:

$$f+g+h \leq m'-2d-2e \overset{(5.79)}{=} m''. \tag{5.84}$$

Hence, sufficiently many levels are also available for the D3C(m'',m'',m''). Altogether, there are enough levels to communicate all $a+b+c+2d+2e+f+g+h$ bits. The application of schemes A to C achieves the upper bounds of the capacity region for the D3C(m,m,m) proving Theorem 5.5. The remaining steps, showing that each corner point of the capacity region is achievable is analogous to [79, Theorems 3 & 4] and omitted here. ■

Note that, by considering the CPCM of the 3-way channel first, we were able to facilitate the analysis of the 3-way channel within the LDCM.

5.3 Degrees of Freedom of the MIMO $M_i \times M_i$ 3-Way Channel

In the following, we investigate MIMO IA on the Gaussian 3-way channel with multiple antennas. The DoF of several unidirectional multi-user interference networks with particular antenna configurations have already been studied thoroughly in the literature (cf. References of [12]). Therein, a particular focus concerns the DoF for MIMO IA with constant channel coefficients. For instance, the DoF of the 2-user MIMO interference channel using zero-forcing are provided in [35], the sum-DoF and the DoF-region of the 2-user MIMO X-channel are considered in [11] and [38], respectively, where MIMO IA was used. The DoF of the general MIMO K-user interference channel with an arbitrary number of antennas at the transmitters and receivers are yet unknown and a full derivation of the remains quite challenging so far.

In this section, we study the DoF of the MIMO 3-way channel with constant channel coefficients. In this first case, we assume an arbitrary number of M_i transmit antennas and also M_i receive antennas at each transceiver T_i. We derive cut-set and genie-aided upper bounds and on the sum-DoF of the MIMO 3-way channel. A MIMO IA and zero-forcing scheme is proposed to show that the derived upper bound is achievable. We observe that the sum-DoF are limited by the strongest channel (the one with the largest rank), and therefore, the sum-DoF are achievable by just letting the two strongest users communicate similar to the single-input single-output (SISO) case [91]. Since this does not serve all users in a fair manner, we propose an alternative scheme which also achieves the sum-DoF upper bound while serving all three users simultaneously. The achievable DoF of this alternative scheme is expressed in terms of a linear programming optimization problem which can be solved by using the well-known simplex method. As in the cases of the CPCM and the LDCM, the considered MIMO 3-way channel is closely related to the MIMO 3-user Y-channel [77], [78].

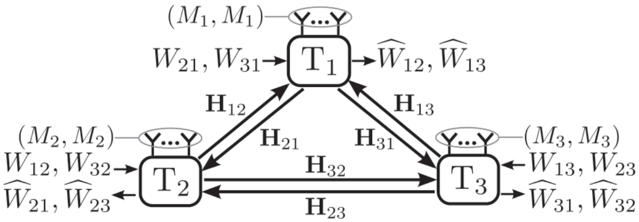


Figure 5.13: The MIMO 3-Way Δ-channel with an equal number of $M_{\mathrm{Tx}_i} = M_i$ transmit antennas and $M_{\mathrm{Rx}_i} = M_i$ receive antennas at each user $i = 1, 2, 3$.

5.3.1 MIMO $M_i \times M_i$ 3-Way Channel

The MIMO 3-way channel comprises three full-duplex users T_i with user-indices i in the set $\mathcal{K} = \{1, 2, 3\}$. A message from T_i to T_j is denoted by W_{ji} and has rate R_{ji} for $i \neq j \in \mathcal{K}$. Each user desires to communicate a message to the two other users. In general, a transceiver T_i is equipped with an arbitrary number of $M_{\mathrm{Tx}_i} \in \mathbb{N}$ transmit antennas and a number of number of $M_{\mathrm{Rx}_i} \in \mathbb{N}$ receive antennas. However in this section, we consider the symmetric special case, where the number of transmit and receive antennas is assumed to be equal to $M_{\mathrm{Tx}_i} = M_{\mathrm{Rx}_i} = M_i$ per T_i, for $i \in \mathcal{K}$. We may assume w.l.o.g. that the number of antennas is ordered among the three users by:

$$M_1 \geq M_2 \geq M_3. \tag{5.85}$$

As defined in Section 2.2, the received signal for the 3-way channel is particularly defined by:

$$\boldsymbol{y}_1(n) = \boldsymbol{H}_{12}\boldsymbol{x}_2(n) + \boldsymbol{H}_{13}\boldsymbol{x}_3(n), \tag{5.86}$$
$$\boldsymbol{y}_2(n) = \boldsymbol{H}_{21}\boldsymbol{x}_1(n) + \boldsymbol{H}_{23}\boldsymbol{x}_3(n), \tag{5.87}$$
$$\boldsymbol{y}_3(n) = \boldsymbol{H}_{31}\boldsymbol{x}_1(n) + \boldsymbol{H}_{32}\boldsymbol{x}_2(n). \tag{5.88}$$

The considered channel coefficients of each $\boldsymbol{H}_{ji}$ are assumed to be constant throughout the whole duration of the transmission and the coefficients are fully known at each user.

Encoding and Decoding Functions

After receiving $\boldsymbol{y}_j(n)$, T_j constructs $\boldsymbol{x}_j(n+1)$ as:

$$\boldsymbol{x}_j(n+1) = \mathcal{E}_{j,n}(W_{ij}, W_{kj}, \boldsymbol{y}_j^n), \tag{5.89}$$

where $\mathcal{E}_{j,n}$ is the encoding function of T_j at time-instant n, and sends $\boldsymbol{x}_j(n+1)$ in the next transmission. After N transmissions, where N is the length of one transmission block (codeword), T_j decodes W_{ji} and W_{jk} as follows:

$$(W_{ji}, W_{jk}) = \mathcal{D}_j(W_{ij}, W_{kj}, \boldsymbol{y}_j^N), \tag{5.90}$$

where $\mathcal{D}_j$ is the decoding function of T_j. We will neglect the time-index n for notational simplicity unless necessary.

5.3.2 Intersection Subspace of Random Matrices

For the derivation of the capacity-achieving schemes, we need to compute the intersection subspaces of the spaces spanned by the channel matrices. This is accomplished by the following lemma[7].

Lemma 5.6. *If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are complex $L \times M_1$ and $L \times M_2$ random matrices, respectively, whose entries are drawn randomly i. i. d., then there exists an intersection subspace of $[\min(M_1, L) + \min(M_2, L) - L]^+$ dimensions between the two column spaces of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, almost surely.*

Proof:

[7] The proof is a slightly generalized version of [77, Lemma 1].

Let an $L \times 1$ vector $\boldsymbol{q}$ lie in $\mathsf{span}(\boldsymbol{A}_1) \cap \mathsf{span}(\boldsymbol{A}_2)$. Then, there exists $\boldsymbol{q}_i \in \mathbb{C}^{M_i \times 1}$, with $i = 1, 2$, such that:

$$\boldsymbol{q} = \boldsymbol{A}_1 \boldsymbol{q}_1 = \boldsymbol{A}_2 \boldsymbol{q}_2. \tag{5.91}$$

In matrix form this yields:

$$\begin{bmatrix} \boldsymbol{I}_{L\times L} & -\boldsymbol{A}_1 & \boldsymbol{0}_{L\times M_2} \\ \boldsymbol{I}_{L\times L} & \boldsymbol{0}_{L\times M_1} & -\boldsymbol{A}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{pmatrix} = \boldsymbol{M}\boldsymbol{x} = \boldsymbol{0}_{2L\times \mathsf{rank}(\boldsymbol{M})}. \tag{5.92}$$

Note that $\mathsf{rank}(\boldsymbol{A}_i) = \min(M_i, L)$ holds, almost surely. We compute the dimension of $\mathsf{span}(\boldsymbol{A}_1) \cap \mathsf{span}(\boldsymbol{A}_2)$ by computing the dimension of the nullity of $\boldsymbol{M}$. Since:

$$\mathsf{rank}(\boldsymbol{M}) = \min(2L, \min(M_1, L) + \min(M_2, L) + L)$$

holds for i.i.d. matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ almost surely, we can conclude with the rank-nullity theorem of linear algebra, that:

$$\begin{aligned} &\mathsf{dim}(\mathsf{null}(\boldsymbol{M})) \\ &= \min(M_1, L) + \min(M_2, L) + L - \mathsf{rank}(\boldsymbol{M}) \\ &= [\min(M_1, L) + \min(M_2, L) - L]^+ \end{aligned} \tag{5.93}$$

holds, almost surely. ■

5.3.3 Main Result

The main result of Section 5.3 is a sum-DoF characterization for the MIMO 3-way channel as provided in the following theorem.

Theorem 5.7. *The DoF of the MIMO 3-way channel with M_i antennas at user T_i, and $M_1 \geq M_2 \geq M_3$, are given by:*

$$d_\Sigma = d_{12} + d_{21} + d_{13} + d_{31} + d_{23} + d_{32} = 2M_2. \tag{5.94}$$

The converse of this theorem is provided in Section 5.4.2 and the achievability in Section 5.4.3. This theorem states that the sum-DoF in this case is given by twice the rank of the channel matrix between T_1 and T_2, which is the channel of largest rank. Therefore, this DoF is achievable by letting these two users communicate while leaving T_3 silent. Albeit this achieves $2M_2$ DoF, it completely excludes T_3 and it does not distribute the resources fairly between the three users. In Section 5.4.3, we provide an alternative scheme which achieves the DoF while maintaining non-zero DoF for all users.

5.3.4 Upper Bounds

Cut-Set Bounds

We begin with considering the cut-set bounds for the MIMO 3-way channel:

$$d_{ji} + d_{ki} \le \min(M_i, M_j + M_k), \tag{5.95}$$
$$d_{ij} + d_{ik} \le \min(M_j + M_k, M_i). \tag{5.96}$$

The right-hand side of (5.95) is the rank of the MIMO channel between T_i and a receiver formed by enabling full cooperation between T_j and T_k, with channel matrix $[\boldsymbol{H}_{ji}^{\mathsf{T}} \boldsymbol{H}_{ki}^{\mathsf{T}}]^{\mathsf{T}}$. A similar interpretation holds for the second bound.

Similar to [78], the cut-set bounds provide bounds on the sum of the DoF of two messages at a time. However, using genie-aided arguments, it is possible to establish bounds on the sum of the DoF of three messages, which are tighter than the cut-set bounds. The key idea is to allow some user to decode one more message in addition to its two desired messages by enhancing this user with some side-information.

Genie-Aided Bounds

Assume every node can obtain its dedicated messages with an arbitrary small probability of error. This means that T_2 for instance can decode its dedicated messages W_{21} and W_{23} reliably from its available information, i. e., from its own transmitted W_{12}, W_{32}, and from its received signal $\boldsymbol{y}_2^N$. Now let us enhance T_2 by providing the message W_{31} as side-information. We also provide T_2 with the correction-noise signal:

$$\tilde{\boldsymbol{z}}_2^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{13} \boldsymbol{H}_{23} \boldsymbol{z}_2^N, \tag{5.97}$$

as side-information[8].

At this point, T_2 knows W_{21} (decoded) and W_{31} (side-information). With W_{21}, W_{31}, T_2 can generate $\boldsymbol{x}_1(1)$. By subtracting $\boldsymbol{H}_{21}\boldsymbol{x}_1(1)$ from $\boldsymbol{y}_2(1)$, and multiplying the result with $\boldsymbol{H}_{23}$, T_2 can recover a noisy observation of $\boldsymbol{x}_3(1)$ given by $\boldsymbol{x}_3(1)+\boldsymbol{H}_{23}\boldsymbol{z}_2(1)$. Next, T_2 multiplies this noisy observation by $\boldsymbol{H}_{13}$, and adds $\boldsymbol{H}_{12}\boldsymbol{x}_2(1)$ and $\tilde{\boldsymbol{z}}_2(1)$ to it to obtain $\boldsymbol{y}_1(1)$. Thus, T_2 obtains the first instance of $\boldsymbol{y}_1^n$. Knowing $\boldsymbol{y}_1(1)$, W_{21} and W_{31}, T_2 can generate $\boldsymbol{x}_1(2)$ (cf. (5.89)). Using $\boldsymbol{x}_1(2)$ again with $\boldsymbol{y}_2(2)$, T_2 can generate $\boldsymbol{y}_1(2)$ and $\boldsymbol{x}_1(3)$. T_2 proceeds this way until all instances (up to the N-th instance) of $\boldsymbol{y}_1^N$ have been generated. Now, having $\boldsymbol{y}_1^N$, W_{21}, and W_{31}, i. e., the same information as T_1, T_2 can decode W_{13} (cf. (5.90)). Therefore, given W_{13} and $\tilde{\boldsymbol{z}}_2^N$ as side-information, T_2 can decode W_{21}, W_{23} and W_{13}. Hence, the DoF of these messages are almost surely upper bounded by:

$$d_{21} + d_{23} + d_{31} \le \mathsf{rank}([\boldsymbol{H}_{21}\, \boldsymbol{H}_{23}]) \tag{5.98}$$
$$= \min(M_2, M_1 + M_3) \overset{(5.85)}{=} M_2. \tag{5.99}$$

We can apply a similar approach to bound $d_{31} + d_{32} + d_{12}$ by M_2. However, in this case, we need to enhance T_3 with $M_2 - M_3$ antennas to make it as strong as T_2. The

[8] $\boldsymbol{H}_{23}$ exists since $\boldsymbol{H}_{23}$ is an $M_2 \times M_3$ matrix with $M_2 \ge M_3$.

effective channel output at T_3 after this enhancement becomes:

$$\tilde{\boldsymbol{y}}_3(n) = \tilde{\boldsymbol{H}}_{31}\boldsymbol{x}_1(n) + \tilde{\boldsymbol{H}}_{32}\boldsymbol{x}_2(n) + \tilde{\boldsymbol{z}}_3(n), \tag{5.100}$$

for $n = 1, ..., N$, where $\tilde{\boldsymbol{H}}_{31}$ and $\tilde{\boldsymbol{H}}_{32}$ are $M_2 \times M_1$ and $M_2 \times M_2$ matrices with rank M_2, respectively, and $\tilde{\boldsymbol{z}}_3$ is a Gaussian noise vector with M_2 dimensions. T_3 can decode W_{31}, W_{32} having $\tilde{\boldsymbol{y}}_3^n$, W_{13}, W_{23}. By providing W_{21} and:

$$\tilde{\boldsymbol{z}}_3^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{12}\tilde{\boldsymbol{H}}_{32}^{-1}\boldsymbol{z}_3^N \tag{5.101}$$

to the enhanced T_3 with M_2 antennas, it can generate $\boldsymbol{x}_1(1)$. We use analogous operations as applied for (5.99) to obtain $\boldsymbol{y}_1^N$ and to decode W_{12}. This leads to the upper bound:

$$\begin{aligned} d_{31} + d_{32} + d_{12} &\leq \mathsf{rank}([\tilde{\boldsymbol{H}}_{31}\, \tilde{\boldsymbol{H}}_{32}]) && (5.102)\\ &= \min(M_2, M_1 + M_2) = M_2, && (5.103)\end{aligned}$$

almost surely. Concluding the converse proof by combining (5.99) and (5.103) yields the sum-DoF upper bound of Theorem 5.7:

$$d_\Sigma = d_{12} + d_{21} + d_{13} + d_{31} + d_{23} + d_{32} \leq 2M_2. \tag{5.104}$$

To achieve the upper bound on the sum-DoF, we propose a beam-forming and zero-forcing scheme using MIMO interference alignment [78] in the following sections.

5.3.5 Achievability

Pre-coding

We consider the receive signal space at T_1 at first. Note that as T_2 and T_3 each have less antennas than T_1, they can not beam-form interference into the null space of T_1. Instead of zero-forcing beam-forming, we use interference alignment. In order to minimize the number of dimensions spanned by the interference caused by T_2 and T_3 at T_1, we align the interference caused by the bidirectional communication between T_2 and T_3 (signals $\boldsymbol{u}_{32}$ and $\boldsymbol{u}_{23}$, respectively) in the intersection subspace of the spaces spanned by the columns of $\boldsymbol{H}_{12}$ and $\boldsymbol{H}_{13}$. From Lemma 5.6, the columns of $\boldsymbol{H}_{12}$ and $\boldsymbol{H}_{13}$ intersect in an $\tilde{M}_1$-dimensional subspace, where $\tilde{M}_1 = [M_2 + M_3 - M_1]^+$. To achieve this alignment, T_2 and T_3 pre-code the signal streams $\boldsymbol{u}_{32} \in \mathbb{C}^{\tilde{d}_{32}}$ and $\boldsymbol{u}_{23} \in \mathbb{C}^{\tilde{d}_{23}}$ with:

$$0 \leq \tilde{d}_{32} = \tilde{d}_{23} \leq \tilde{M}_1 \tag{5.105}$$

dimensions, into $\boldsymbol{V}_{32}\boldsymbol{u}_{32}$ and $\boldsymbol{V}_{23}\boldsymbol{u}_{23}$, respectively, where the beam-forming matrices $\boldsymbol{V}_{32} \in \mathbb{C}^{M_2 \times \tilde{d}_{32}}$ and $\boldsymbol{V}_{23} \in \mathbb{C}^{M_3 \times \tilde{d}_{32}}$ satisfy the following alignment at T_1:

$$\mathsf{span}(\boldsymbol{H}_{13}\boldsymbol{V}_{23}) = \mathsf{span}(\boldsymbol{H}_{12}\boldsymbol{V}_{32}). \tag{5.106}$$

This accounts for a total of $2\tilde{d}_{32}$ streams that can be exchanged by T_2 and T_3 while causing interference in only $\tilde{d}_{32}$ dimensions at T_1.

Now, we consider the receive signal space at T_2. As T_1 has more antennas than T_2, T_1 can send a signal $\bar{\boldsymbol{u}}_{31} \in \mathbb{C}^{\bar{d}_{31}}$ to T_3 in the null space of $\boldsymbol{H}_{21}$. The maximal number of such streams that can be beam-formed to this null space is bounded by $\min(M_1 - M_2, M_3)$. Thus, T_1 sends streams of:

$$0 \le \bar{d}_{31} \le \min(M_1 - M_2, M_3) \tag{5.107}$$

dimensions beam-formed into the null space of $\boldsymbol{H}_{21}$. To realize this, T_1 designs a zero-forcing beam-forming matrix $\bar{\boldsymbol{V}}_{31} \in \mathbb{C}^{M_1 \times \bar{d}_{31}}$ that satisfies:

$$\boldsymbol{H}_{21}\bar{\boldsymbol{V}}_{31} = \mathbf{0}_{M_2 \times \bar{d}_{31}}, \tag{5.108}$$

and pre-codes $\bar{\boldsymbol{u}}_{31}$ by $\bar{\boldsymbol{V}}_{31}\bar{\boldsymbol{u}}_{31}$. The remaining streams sent from T_1 to T_3 (if any) can be aligned to the streams sent from T_3 to T_1 within the receive signal space of T_2. This alignment is possible since the columns of $\boldsymbol{H}_{21}$ and $\boldsymbol{H}_{23}$ intersect in an M_3-dimensional subspace as given by Lemma 5.6. To this end, T_1 and T_3 construct $\tilde{\boldsymbol{V}}_{31}\tilde{\boldsymbol{u}}_{31}$ and $\boldsymbol{V}_{13}\boldsymbol{u}_{13}$, respectively, where $\tilde{\boldsymbol{u}}_{31} \in \mathbb{C}^{\tilde{d}_{31}}$ and $\boldsymbol{u}_{13} \in \mathbb{C}^{\tilde{d}_{13}}$ have:

$$0 \le \tilde{d}_{31} = \tilde{d}_{13} \le M_3 \tag{5.109}$$

dimensions, and where the beam-forming matrices defined by $\boldsymbol{V}_{13} \in \mathbb{C}^{M_3 \times \tilde{d}_{31}}$, and $\tilde{\boldsymbol{V}}_{31} \in \mathbb{C}^{M_1 \times \tilde{d}_{31}}$ satisfy:

$$\mathsf{span}(\boldsymbol{H}_{23}\boldsymbol{V}_{13}) = \mathsf{span}(\boldsymbol{H}_{21}\tilde{\boldsymbol{V}}_{31}). \tag{5.110}$$

The aligned interference of $\tilde{\boldsymbol{u}}_{31}$ and $\boldsymbol{u}_{13}$ occupies $\tilde{d}_{31} \le M_3$ dimensions at the receive signal space of T_2.

Considering the interference space at T_3, we see that T_3 has less antennas than T_1 and than T_2. Thus, T_1 beam-forms a signal $\bar{\boldsymbol{u}}_{21} \in \mathbb{C}^{\bar{d}_{21}}$ into the null space of $\boldsymbol{H}_{31}$ of size $\min(M_2, M_1 - M_3)$, which requires:

$$0 \le \bar{d}_{21} \le \min(M_2, M_1 - M_3) \tag{5.111}$$

dimensions. This is accomplished by designing a zero-forcing beam-forming matrix $\bar{\boldsymbol{V}}_{21} \in \mathbb{C}^{M_1 \times \bar{d}_{21}}$ such that:

$$\boldsymbol{H}_{31}\bar{\boldsymbol{V}}_{21} = \mathbf{0}_{M_3 \times \bar{d}_{21}}, \tag{5.112}$$

and by pre-coding $\bar{\boldsymbol{u}}_{21}$ with $\bar{\boldsymbol{V}}_{21}\bar{\boldsymbol{u}}_{21}$. Then, T_2 beam-forms $\bar{\boldsymbol{u}}_{12} \in \mathbb{C}^{\bar{d}_{12}}$ into the null space at T_3 of size $M_2 - M_3$, where:

$$0 \le \bar{d}_{12} \le M_2 - M_3. \tag{5.113}$$

To realize this, we design a zero-forcing beam-forming matrix $\bar{\boldsymbol{V}}_{12} \in \mathbb{C}^{M_2 \times \bar{d}_{12}}$ such that:

$$\boldsymbol{H}_{32}\bar{\boldsymbol{V}}_{12} = \mathbf{0}_{M_3 \times \bar{d}_{12}}, \tag{5.114}$$

and pre-code $\bar{\boldsymbol{u}}_{12}$ by $\bar{\boldsymbol{V}}_{12}\bar{\boldsymbol{u}}_{12}$. The remaining streams from T_1 to T_2 and vice versa (if any) are aligned at T_3. The spaces spanned by $\boldsymbol{H}_{31}$ and $\boldsymbol{H}_{32}$ intersect in M_3 dimensions as given by Lemma 5.6. We choose the beam-forming matrices $\tilde{\boldsymbol{V}}_{21} \in \mathbb{C}^{M_1 \times \tilde{d}_{21}}$ and $\tilde{\boldsymbol{V}}_{12} \in \mathbb{C}^{M_2 \times \tilde{d}_{21}}$ such that:

$$\mathsf{span}(\boldsymbol{H}_{32}\tilde{\boldsymbol{V}}_{12}) = \mathsf{span}(\boldsymbol{H}_{31}\tilde{\boldsymbol{V}}_{21}), \tag{5.115}$$

and use them to pre-code $\tilde{\boldsymbol{u}}_{21}$ and $\tilde{\boldsymbol{u}}_{12}$ with:

$$0 \le \tilde{d}_{21} = \tilde{d}_{12} \le M_3 \tag{5.116}$$

dimensions into $\tilde{\boldsymbol{V}}_{21}\tilde{\boldsymbol{u}}_{21}$ and $\tilde{\boldsymbol{V}}_{12}\tilde{\boldsymbol{u}}_{12}$.

Finally, the transmitters send the following signals:

$$\boldsymbol{x}_1 = [\tilde{\boldsymbol{V}}_{21}\ \bar{\boldsymbol{V}}_{21}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{21}\\ \bar{\boldsymbol{u}}_{21}\end{smallmatrix}\right] + [\tilde{\boldsymbol{V}}_{31}\ \bar{\boldsymbol{V}}_{31}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{31}\\ \bar{\boldsymbol{u}}_{31}\end{smallmatrix}\right], \tag{5.117}$$

$$\boldsymbol{x}_2 = [\tilde{\boldsymbol{V}}_{12}\ \bar{\boldsymbol{V}}_{12}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{12}\\ \bar{\boldsymbol{u}}_{12}\end{smallmatrix}\right] + \boldsymbol{V}_{32}\boldsymbol{u}_{32}, \tag{5.118}$$

$$\boldsymbol{x}_3 = \boldsymbol{V}_{13}\boldsymbol{u}_{13} + \boldsymbol{V}_{23}\boldsymbol{u}_{23}. \tag{5.119}$$

In total, T_1 sends $d_{21} = \tilde{d}_{21} + \bar{d}_{21}$ and $d_{31} = \tilde{d}_{31} + \bar{d}_{31}$ streams to T_2 and T_3, respectively, T_2 sends $d_{12} = \tilde{d}_{12} + \bar{d}_{12}$ and $d_{32} = \tilde{d}_{32}$ streams to T_1 and T_3, respectively, and T_3 sends $d_{13} = \tilde{d}_{12}$ and $d_{23} = \tilde{d}_{23}$ streams to T_1 and T_2, respectively.

Post-coding

The received signal at T_1 can be written as:

$$\boldsymbol{y}_1 = \boldsymbol{H}_{12}[\tilde{\boldsymbol{V}}_{12}\ \bar{\boldsymbol{V}}_{12}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{12}\\ \bar{\boldsymbol{u}}_{12}\end{smallmatrix}\right] + [\boldsymbol{H}_{12}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{H}_{13}\boldsymbol{V}_{23}\boldsymbol{u}_{23}] + \boldsymbol{H}_{13}\boldsymbol{V}_{13}\boldsymbol{u}_{13} + \boldsymbol{z}_1. \tag{5.120}$$

The desired signals from T_2 occupy $\tilde{d}_{21} + \bar{d}_{21}$ dimensions. The aligned interference in the second part of the summation, i. e., $\boldsymbol{H}_{12}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{H}_{13}\boldsymbol{V}_{23}\boldsymbol{u}_{23}$, occupies $\tilde{d}_{32}$ dimensions, and the desired signal from T_3 occupies $\tilde{d}_{13}$ dimensions. The desired signals can be resolved from the interference as long as they are linearly independent of the interference and also among each other. Namely, the columns of the following $M_1 \times (\tilde{d}_{12} + \bar{d}_{12} + \tilde{d}_{32} + \tilde{d}_{13})$ matrix must be linearly independent:

$$\left[\boldsymbol{H}_{12}\tilde{\boldsymbol{V}}_{12}\ \boldsymbol{H}_{12}\bar{\boldsymbol{V}}_{12}\ \boldsymbol{H}_{12}\boldsymbol{V}_{32}\ \boldsymbol{H}_{13}\boldsymbol{V}_{13}\right], \tag{5.121}$$

which requires:

$$0 \le \tilde{d}_{12} + \bar{d}_{12} + \tilde{d}_{32} + \tilde{d}_{13} \le M_1. \tag{5.122}$$

Under this condition, this linear independence can be guaranteed (almost surely) by designing $\bar{\boldsymbol{V}}_{12}$ according to (5.114), and choosing $\tilde{\boldsymbol{V}}_{12}$, $\boldsymbol{V}_{32}$, and $\boldsymbol{V}_{13}$ randomly.

Given this linear independence, T_1 can use zero-forcing matrices $\boldsymbol{N}_{12}$ and $\boldsymbol{N}_{13}$ of $d_{12} \times M_1$ and $d_{13} \times M_1$ dimensions, to zero-force the interference and to separate the two dedicated information signals. These zero-forcing matrices must satisfy:

$$\boldsymbol{N}_{12}\boldsymbol{H}_{13}(\boldsymbol{V}_{13} + \boldsymbol{V}_{23}) = \boldsymbol{0}_{d_{12}\times(d_{13}+d_{23})}, \tag{5.123}$$

$$\boldsymbol{N}_{13}\boldsymbol{H}_{12}(\tilde{\boldsymbol{V}}_{12} + \bar{\boldsymbol{V}}_{12} + \boldsymbol{V}_{32}) = \boldsymbol{0}_{d_{13}\times(d_{12}+d_{32})}. \tag{5.124}$$

Note that by zero-forcing $\boldsymbol{H}_{13}\boldsymbol{V}_{23}$, also $\boldsymbol{H}_{12}\boldsymbol{V}_{32}$ is zero-forced (and vice-versa) by (5.106). By using the proposed null-space beam-forming and zero-forcing, receiver T_1 obtains:

$$\boldsymbol{N}_{12}\boldsymbol{y}_1 = \boldsymbol{N}_{12}\boldsymbol{H}_{12}(\tilde{\boldsymbol{V}}_{12}\tilde{\boldsymbol{u}}_{12} + \bar{\boldsymbol{V}}_{12}\bar{\boldsymbol{u}}_{12}) + \boldsymbol{N}_{12}\boldsymbol{z}_1, \tag{5.125}$$

$$\boldsymbol{N}_{13}\boldsymbol{y}_1 = \boldsymbol{N}_{13}\boldsymbol{H}_{13}\boldsymbol{V}_{13}\boldsymbol{u}_{13} + \boldsymbol{N}_{13}\boldsymbol{z}_1. \tag{5.126}$$

Thus, T_1 recovers d_{12} linearly independent noisy observations of $\tilde{\boldsymbol{u}}_{12}$ and $\bar{\boldsymbol{u}}_{12}$, and also d_{13} linearly independent noisy observations of $\boldsymbol{u}_{13}$ as $\boldsymbol{N}_1 = [\boldsymbol{N}_{12}^\mathsf{T}\boldsymbol{N}_{13}^\mathsf{T}]^\mathsf{T}$ has sufficient row rank $d_{12}+d_{13}$ almost surely. Thus T_1 can decode all dedicated signals and achieves a number of $d_{12}+d_{13}$ DoF.

On the receiver side of T_2, we have:

$$\boldsymbol{y}_2 = \boldsymbol{H}_{21}[\tilde{\boldsymbol{V}}_{21}\ \bar{\boldsymbol{V}}_{21}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{21}\\ \bar{\boldsymbol{u}}_{21}\end{smallmatrix}\right] + [\boldsymbol{H}_{21}\tilde{\boldsymbol{V}}_{31}\tilde{\boldsymbol{u}}_{31} + \boldsymbol{H}_{23}\boldsymbol{V}_{13}\boldsymbol{u}_{13}]\boldsymbol{H}_{23}\boldsymbol{V}_{23}\boldsymbol{u}_{23} + \boldsymbol{z}_2. \tag{5.127}$$

Note that $\bar{\boldsymbol{u}}_{31}$ is not observed by T_2 due to (5.108). Similarly to T_1, we need the following constraint to guarantee the linear independence of the desired signals and the interference:

$$0 \le \tilde{d}_{21} + \bar{d}_{21} + \tilde{d}_{31} + \tilde{d}_{23} \le M_2. \tag{5.128}$$

We use zero-forcing matrices $\boldsymbol{N}_{21}$ and $\boldsymbol{N}_{23}$ of $d_{21}\times M_2$ and $d_{23}\times M_2$ dimensions, respectively, satisfying:

$$\boldsymbol{N}_{21}\boldsymbol{H}_{23}(\boldsymbol{V}_{23} + \boldsymbol{V}_{13}) = \boldsymbol{0}_{d_{21}\times(d_{23}+d_{13})}, \tag{5.129}$$

$$\boldsymbol{N}_{23}\boldsymbol{H}_{21}(\tilde{\boldsymbol{V}}_{21} + \bar{\boldsymbol{V}}_{21} + \tilde{\boldsymbol{V}}_{31}) = \boldsymbol{0}_{d_{23}\times(d_{21}+\tilde{d}_{31})}, \tag{5.130}$$

to zero-force the interference and to separate the two dedicated information signals. By zero-forcing $\boldsymbol{H}_{23}\boldsymbol{V}_{13}$, also $\boldsymbol{H}_{21}\tilde{\boldsymbol{V}}_{31}$ is zero-forced (and vice-versa) by (5.110). With this scheme, receiver T_2 obtains:

$$\boldsymbol{N}_{21}\boldsymbol{y}_2 = \boldsymbol{N}_{21}\boldsymbol{H}_{21}(\tilde{\boldsymbol{V}}_{21}\tilde{\boldsymbol{u}}_{21} + \bar{\boldsymbol{V}}_{21}\bar{\boldsymbol{u}}_{21}) + \boldsymbol{N}_{21}\boldsymbol{z}_2, \tag{5.131}$$

$$\boldsymbol{N}_{23}\boldsymbol{y}_2 = \boldsymbol{N}_{23}\boldsymbol{H}_{23}\boldsymbol{V}_{23}\boldsymbol{u}_{23} + \boldsymbol{N}_{23}\boldsymbol{z}_2. \tag{5.132}$$

T_2 recovers d_{21} linearly independent noisy observations of $\tilde{\boldsymbol{u}}_{21}$ and $\bar{\boldsymbol{u}}_{21}$, and d_{23} linearly independent noisy observations of $\boldsymbol{u}_{23}$ from $\boldsymbol{y}_2$ since $\boldsymbol{N}_2 = [\boldsymbol{N}_{21}^\mathsf{T}\boldsymbol{N}_{23}^\mathsf{T}]^\mathsf{T}$ has sufficient row rank $d_{21}+d_{23}$ almost surely. Hence, T_2 achieves a number of $d_{21}+d_{23}$ DoF.

On the receiver side of T_3, we have:

$$\boldsymbol{y}_3 = \boldsymbol{H}_{31}[\tilde{\boldsymbol{V}}_{31}\ \bar{\boldsymbol{V}}_{31}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{31}\\ \bar{\boldsymbol{u}}_{31}\end{smallmatrix}\right] + [\boldsymbol{H}_{31}\tilde{\boldsymbol{V}}_{21}\tilde{\boldsymbol{u}}_{21} + \boldsymbol{H}_{32}\tilde{\boldsymbol{V}}_{12}\tilde{\boldsymbol{u}}_{12}] + \boldsymbol{H}_{32}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{z}_3. \tag{5.133}$$

At T_3, the signals $\bar{\boldsymbol{u}}_{21}$ and $\bar{\boldsymbol{u}}_{21}$ are not observed due to (5.112) and (5.114). We need the following constraint to guarantee the linear independence of the desired signals and the interference:

$$0 \le \tilde{d}_{31} + \bar{d}_{31} + \tilde{d}_{21} + \tilde{d}_{32} \le M_3. \tag{5.134}$$

We use zero-forcing matrices $\boldsymbol{N}_{31}$ and $\boldsymbol{N}_{32}$ of dimensions $d_{31} \times M_3$ and $d_{32} \times M_3$, satisfying:

$$\boldsymbol{N}_{31}\boldsymbol{H}_{32}(\boldsymbol{V}_{32} + \tilde{\boldsymbol{V}}_{12}) = \mathbf{0}_{d_{31}\times(d_{32}+\tilde{d}_{12})}, \tag{5.135}$$

$$\boldsymbol{N}_{32}\boldsymbol{H}_{31}(\tilde{\boldsymbol{V}}_{31} + \bar{\boldsymbol{V}}_{31} + \tilde{\boldsymbol{V}}_{21}) = \mathbf{0}_{d_{23}\times(d_{31}+\tilde{d}_{21})}, \tag{5.136}$$

to zero-force the interference space and to separate the two dedicated information signals. With this scheme, receiver T_3 obtains:

$$\boldsymbol{N}_{31}\boldsymbol{y}_3 = \boldsymbol{N}_{31}\boldsymbol{H}_{31}(\tilde{\boldsymbol{V}}_{31}\tilde{\boldsymbol{u}}_{31} + \bar{\boldsymbol{V}}_{31}\bar{\boldsymbol{u}}_{31}) + \boldsymbol{N}_{31}\boldsymbol{z}_3, \tag{5.137}$$

$$\boldsymbol{N}_{32}\boldsymbol{y}_3 = \boldsymbol{N}_{32}\boldsymbol{H}_{32}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{N}_{32}\boldsymbol{z}_3. \tag{5.138}$$

Thus, T_3 can recover d_{31} linearly independent noisy observations of $\tilde{\boldsymbol{u}}_{31}$ and $\bar{\boldsymbol{u}}_{31}$, and d_{32} linearly independent noisy observations of $\boldsymbol{u}_{32}$ from $\boldsymbol{y}_3$ since $\boldsymbol{N}_3 = [\boldsymbol{N}_{31}^{\mathsf{T}}\boldsymbol{N}_{32}^{\mathsf{T}}]^{\mathsf{T}}$ has sufficient row rank $d_{31} + d_{32}$ almost surely. Hence, T_2 can decode its dedicated signals and achieves $d_{31} + d_{32}$ DoF.

Assembling all constraints on the achievable DoF, yields:

$$\begin{aligned}
\tilde{d}_{32} = \tilde{d}_{23} &\leq [M_2 + M_3 - M_1]^+, \\
\bar{d}_{31} &\leq \min(M_3, M_1 - M_2), \\
\bar{d}_{21} &\leq \min(M_2, M_1 - M_3), \\
\bar{d}_{12} &\leq M_2 - M_3, \\
\tilde{d}_{12} + \bar{d}_{12} + \tilde{d}_{32} + \tilde{d}_{13} &\leq M_1, \\
\tilde{d}_{21} + \bar{d}_{21} + \tilde{d}_{31} + \tilde{d}_{23} &\leq M_2, \\
\tilde{d}_{31} + \bar{d}_{31} + \tilde{d}_{21} + \tilde{d}_{32} &\leq M_3.
\end{aligned}$$

Note that real-valued DoF can be approximated by using signal-extensions over multiple time-slots [38, 78]. By maximizing d_Σ subject to these non-negative constraints, we get the maximum achievable sum-DoF of this scheme. This maximization is a linear optimization problem which can be solved by using the simplex method. The maximization yields a sum-DoF of $2M_2$. To verify this, we set:

$$\tilde{d}_{32} = \tilde{d}_{23} = (M_2 + M_3 - M_1)^+, \tag{5.139}$$

$$\bar{d}_{31} = \min(M_3, M_1 - M_2), \tag{5.140}$$

$$\bar{d}_{21} = \min(M_2, M_1 - M_3), \tag{5.141}$$

$$\bar{d}_{12} = M_2 - M_3. \tag{5.142}$$

This allocation satisfies all the DoF constraints above, and leads to:

$$d_\Sigma = 2\tilde{d}_{32} + \bar{d}_{31} + \bar{d}_{21} + \bar{d}_{12} = 2M_2.$$

5.4 Symmetric Degrees of Freedom of the MIMO $M_{\mathrm{Tx}} \times M_{\mathrm{Rx}}$ 3-Way Channel

In the previous section, it has been shown that the DoF of the $M_i \times M_i$ 3-way channel are limited by M_2, i.e., the rank of the strongest subchannel. In this section, we investigate how the DoF are limited when using a symmetrical number of M_{Tx} transmit and M_{Rx} receive antennas. For multi-way conferencing situations with eminently high and almost symmetric rate demands, as in video conferences for instance, using such homogeneously equipped devices is clearly beneficial when compared to heterogeneous devices. Another particular gain from this homogeneous setup is that the symmetric DoF allocation providing complete fairness among all users is sum-DoF optimal.

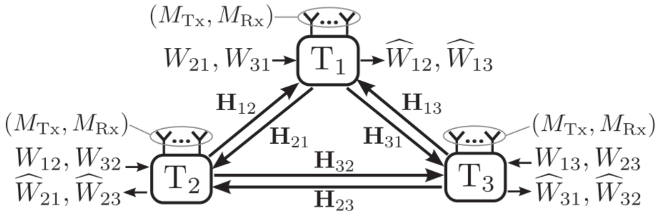


Figure 5.14: The homogeneous MIMO 3-way channel (or Δ-channel) with M_{Tx} transmit and M_{Rx} receive antennas at each user T_i, with $i = 1, 2, 3$.

5.4.1 MIMO $M_{\mathrm{Tx}} \times M_{\mathrm{Rx}}$ 3-Way Channel

We study the DoF of the homogeneous $M_{\mathrm{Tx}} \times M_{\mathrm{Rx}}$ MIMO 3-way channel with constant channel coefficients and with a number of $M_{\mathrm{Tx}_i} = M_{\mathrm{Tx}}$ transmit antennas at each transceiver and $M_{\mathrm{Rx}_i} = M_{\mathrm{Rx}}$ receive antennas.

The system model of the $M_{\mathrm{Tx}} \times M_{\mathrm{Rx}}$ 3-way channel is closely related to the model defined in Section 5.3.1. The input-output relationship is analogously defined by (5.86) to (5.88), but for the given homogeneous antenna configurations. The encoding and decoding functions defined in Section 5.3.1 also carry over to this channel model.

5.4.2 Upper Bounds

Cut-set bounds

We begin with considering the cut-set bounds for the MIMO 3-way channel:

$$d_{ji} + d_{ki} \le \min(M_{\mathrm{Tx}}, 2M_{\mathrm{Rx}}), \tag{5.143}$$

$$d_{ij} + d_{ik} \le \min(2M_{\mathrm{Tx}}, M_{\mathrm{Rx}}). \tag{5.144}$$

The right-hand side of (5.143) is the rank of the MIMO channel between T_i and a receiver formed by enabling full cooperation between T_j and T_k, with channel matrix

$[\boldsymbol{H}_{ji}^{\mathsf{T}}\boldsymbol{H}_{ki}^{\mathsf{T}}]^{\mathsf{T}}$. A similar interpretation holds for the second bound. Combining (5.143) and (5.144) provides the sum-DoF bound:

$$d_{\Sigma} \leq \min(3M_{\mathrm{Tx}}, 3M_{\mathrm{Rx}}). \tag{5.145}$$

Genie-aided bounds

We first assume that $M_{\mathrm{Rx}} \geq M_{\mathrm{Tx}}$. Assume every node can obtain its dedicated messages with an arbitrary small probability of error. Hence, T_2 can decode W_{21}, W_{23} reliably from its available information, i. e., from $\boldsymbol{y}_2^N$, W_{12} and W_{32}, as shown in (5.90). Furthermore, we provide W_{31} to T_2 as side-information. We also provide T_2 with the correction-noise signal:

$$\tilde{\boldsymbol{z}}_2^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{13}\boldsymbol{H}_{23}^{\dagger}\boldsymbol{z}_2^N, \tag{5.146}$$

as side-information[9]. Now, T_2 knows its decoded W_{21} and W_{31} by side-information. With W_{21}, W_{31}, user T_2 can generate $\boldsymbol{x}_1(1)$. By subtracting $\boldsymbol{H}_{21}\boldsymbol{x}_1(1)$ from $\boldsymbol{y}_2(1)$, and multiplying the result with $\boldsymbol{H}_{23}^{\dagger}$, T_2 can recover a noisy observation of $\boldsymbol{x}_3(1)$ given by $\boldsymbol{x}_3(1) + \boldsymbol{H}_{23}^{\dagger}\boldsymbol{z}_2(1)$. Next, T_2 multiplies this noisy observation by $\boldsymbol{H}_{13}$, and adds $\boldsymbol{H}_{12}\boldsymbol{x}_2(1)$ and $\tilde{\boldsymbol{z}}_2(1)$ to it to obtain $\boldsymbol{y}_1(1)$. Thus, T_2 obtains the first instance of $\boldsymbol{y}_1^N$. Knowing $\boldsymbol{y}_1(1)$, W_{21} and W_{31}, T_2 can generate $\boldsymbol{x}_1(2)$ (cf. (5.89)). Using $\boldsymbol{x}_1(2)$ again with $\boldsymbol{y}_2(2)$, T_2 can generate $\boldsymbol{y}_1(2)$ and $\boldsymbol{x}_1(3)$. T_2 proceeds this way until all instances (up to the N- th instance) of $\boldsymbol{y}_1^N$ have been generated. Now, having $\boldsymbol{y}_1^N$, W_{21}, and W_{31}, i. e., the same information as T_1, T_2 can decode W_{13} (cf. (5.90)). Therefore, given W_{13} and $\tilde{\boldsymbol{z}}_2^N$ as side-information, T_2 can decode W_{21}, W_{23} and W_{13}. Hence, the DoF of these messages are almost surely upper bounded by:

$$\begin{aligned} d_{21} + d_{23} + d_{31} &\leq \mathsf{rank}([\boldsymbol{H}_{21}\ \boldsymbol{H}_{23}]) && (5.147)\\ &= \min(M_{\mathrm{Rx}}, 2M_{\mathrm{Tx}}). && (5.148)\end{aligned}$$

We can apply a similar approach to bound:

$$\begin{aligned} d_{31} + d_{32} + d_{12} &\leq \mathsf{rank}([\boldsymbol{H}_{31}\ \boldsymbol{H}_{32}]) && (5.149)\\ &= \min(M_{\mathrm{Rx}}, 2M_{\mathrm{Tx}}), && (5.150)\end{aligned}$$

by providing W_{21} and the correction noise-signal:

$$\tilde{\boldsymbol{z}}_3^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{12}\boldsymbol{H}_{32}^{\dagger}\boldsymbol{z}_3^N \tag{5.151}$$

to T_3. As a result, T_3 can construct $\boldsymbol{y}_1^N$ and decode W_{12} reliably. Combining (5.148) and (5.150), bounds the sum-DoF to:

$$d_{\Sigma} \leq \min(2M_{\mathrm{Rx}}, 4M_{\mathrm{Tx}}). \tag{5.152}$$

For the contrary case, we assume that $M_{\mathrm{Rx}} < M_{\mathrm{Tx}}$ holds. We enhance the number of receive antennas at all receivers to $\mathring{M}_{\mathrm{Rx}} = M_{\mathrm{Tx}}$. The effective channel output at T_3 becomes:

$$\mathring{\boldsymbol{y}}_3(n) = \mathring{\boldsymbol{H}}_{31}\boldsymbol{x}_1(n) + \mathring{\boldsymbol{H}}_{32}\boldsymbol{x}_2(n) + \mathring{\boldsymbol{z}}_3(n), \tag{5.153}$$

[9]The pseudo-inverse $\boldsymbol{H}_{23}^{\dagger}$ exists almost surely, since $\boldsymbol{H}_{23}$ is an $M_{\mathrm{Rx}} \times M_{\mathrm{Tx}}$ matrix with $M_{\mathrm{Rx}} \geq M_{\mathrm{Tx}}$.

with the extended $M_{\mathrm{Tx}} \times M_{\mathrm{Tx}}$ matrices[10] $\mathring{\boldsymbol{H}}_{31}$, $\mathring{\boldsymbol{H}}_{32}$, and the extended $M_{\mathrm{Tx}} \times 1$ noise vector $\mathring{\boldsymbol{z}}_3(n)$. We can apply the upper bounds derived in (5.148) and (5.150) now, leading to:

$$d_{21} + d_{23} + d_{31} \leq M_{\mathrm{Tx}}, \tag{5.154}$$

$$d_{31} + d_{32} + d_{12} \leq M_{\mathrm{Tx}}, \tag{5.155}$$

$$d_\Sigma = 2\min(\mathring{M}_{\mathrm{Rx}}, 2M_{\mathrm{Tx}}) = \min(2M_{\mathrm{Tx}}, 4M_{\mathrm{Tx}}) = 2M_{\mathrm{Tx}}. \tag{5.156}$$

Combining these bounds with the cut-set upper bounds yields:

$$\begin{aligned} d_\Sigma &\leq \min(2M_{\mathrm{Rx}}, 4M_{\mathrm{Tx}}, 3M_{\mathrm{Rx}}, 3M_{\mathrm{Tx}}) \\ &= \min(2M_{\mathrm{Rx}}, 3M_{\mathrm{Tx}}), \text{ if } M_{\mathrm{Tx}} \leq M_{\mathrm{Rx}}, \end{aligned} \tag{5.157}$$

$$\begin{aligned} d_\Sigma &\leq \min(2M_{\mathrm{Tx}}, 3M_{\mathrm{Rx}}, 3M_{\mathrm{Tx}}) \\ &= \min(2M_{\mathrm{Tx}}, 3M_{\mathrm{Rx}}), \text{ if } M_{\mathrm{Tx}} > M_{\mathrm{Rx}}. \end{aligned} \tag{5.158}$$

Theorem 5.8. *The DoF of the $M_{\mathrm{Tx}} \times M_{\mathrm{Rx}}$ MIMO 3-way channel with M_{Tx} transmit antennas and M_{Rx} receive antennas at each user T_i are:*

$$d_\Sigma = \begin{cases} \min(2M_{\mathrm{Rx}}, 3M_{\mathrm{Tx}}), & \text{if } M_{\mathrm{Tx}} \leq M_{\mathrm{Rx}}, \\ \min(2M_{\mathrm{Tx}}, 3M_{\mathrm{Rx}}), & \text{if } M_{\mathrm{Tx}} > M_{\mathrm{Rx}}. \end{cases} \tag{5.159}$$

5.4.3 Achievability

The following two communication schemes provide achievability of the upper bounds in Theorem 5.8. Note that, symbol-extensions on the constant MIMO channels over multiple time-slots are used to achieve non-integer DoF per user, cf. [38], [78]. We discern two cases: either $M_{\mathrm{Tx}} \leq M_{\mathrm{Rx}}$ or $M_{\mathrm{Tx}} > M_{\mathrm{Rx}}$ holds.

Case $M_{\mathrm{Tx}} \leq M_{\mathrm{Rx}}$ with $d_\Sigma = 3M_{\mathrm{Tx}}$

The dominant term in (5.159) yields $3M_{\mathrm{Tx}}$ if $3M_{\mathrm{Tx}} \leq 2M_{\mathrm{Rx}}$ holds. We use the following symmetric DoF allocation:

$$d = d_{ij} = d_{ji}. \tag{5.160}$$

We further decompose the symmetric DoF d further for IA (tilde-notation, $\tilde{d}$) and for beam-forming (bar-notation, $\bar{d}$):

$$\bar{d} = \bar{d}_{ij} = \bar{d}_{ji}, \tag{5.161}$$

$$\tilde{d} = \tilde{d}_{ij} = \tilde{d}_{ji}, \tag{5.162}$$

$$d = \bar{d}_i + \tilde{d}_i. \tag{5.163}$$

In other words, we demand that bidirectional signals pairwise occupy the same number of DoF. According to the assumptions on M_{Tx} and M_{Rx}, the following bounds must hold:

$$0 \leq 2d \leq M_{\mathrm{Tx}}, \tag{5.164}$$

$$0 \leq 3d \leq \min(M_{\mathrm{Rx}}, 2M_{\mathrm{Tx}}), \tag{5.165}$$

[10] The inverses exist almost surely.

so that all upper bounds provided in Section 5.4.2 are satisfied.

Messages W_{ji} are encoded into complex-valued symbol streams $\tilde{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\tilde{d}\times 1}$ and $\bar{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\bar{d}\times 1}$. These symbol streams are pre-coded at the transmitters and post-coded at the receivers, so that the proposed sum-DoF are achieved. For pre-coding, we use beam-forming matrices $\tilde{\boldsymbol{V}}_{ji} \in \mathbb{C}^{M_{\mathrm{Tx}}\times \tilde{d}}$ and $\bar{\boldsymbol{V}}_{ji} \in \mathbb{C}^{M_{\mathrm{Tx}}\times \bar{d}}$. Transmit signals $\boldsymbol{x}_i$ are constructed from the pre-coded symbol streams as:

$$\boldsymbol{x}_i = [\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ji}\\ \bar{\boldsymbol{u}}_{ji}\end{smallmatrix}\right] + [\tilde{\boldsymbol{V}}_{ki}\ \bar{\boldsymbol{V}}_{ki}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ki}\\ \bar{\boldsymbol{u}}_{ki}\end{smallmatrix}\right]. \tag{5.166}$$

First, we consider the intersection space of the two incident subchannels at the receiver of T_j. The number of dimensions for $\mathsf{span}(\boldsymbol{H}_{ji}) \cap \mathsf{span}(\boldsymbol{H}_{jk})$ is computed by Lemma 5.6 as given in the appendix. It has $0 \le [2M_{\mathrm{Tx}} - M_{\mathrm{Rx}}]^+ \le \frac{1}{3}M_{\mathrm{Rx}}$ dimensions, since $3M_{\mathrm{Tx}} \le 2M_{\mathrm{Rx}}$. We fix:

$$\tilde{d} = [2M_{\mathrm{Tx}} - M_{\mathrm{Rx}}]^+, \tag{5.167}$$

and design $\tilde{\boldsymbol{V}}_{ji}$ such that the two dedicated signals remain distinct, while the undesired interference is aligned at each undesired receiver:

$$\mathsf{span}(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}) = \mathsf{span}(\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}). \tag{5.168}$$

After this first part of pre-coding, a number of:

$$\bar{M}_{\mathrm{Tx}} = M_{\mathrm{Tx}} - 2\tilde{d} \ge 0, \tag{5.169}$$

$$\bar{M}_{\mathrm{Rx}} = M_{\mathrm{Rx}} - 3\tilde{d} \ge 0, \tag{5.170}$$

transmit and receive dimensions remain available at each user, respectively. Since the complete intersection space is already consumed by IA, we have $[2\bar{M}_{\mathrm{Tx}} - \bar{M}_{\mathrm{Rx}}]^+ = 0$ and hence, $2\bar{M}_{\mathrm{Tx}} \le \bar{M}_{\mathrm{Rx}}$ holds. The remaining DoF are allocated by:

$$\bar{d} = \tfrac{1}{2}\bar{M}_{\mathrm{Tx}}, \tag{5.171}$$

for all $k \in \mathcal{K}$. The beam-forming matrices $\bar{\boldsymbol{V}}_{ji}$ and $\bar{\boldsymbol{V}}_{jk}$ are chosen such that the two signals $\boldsymbol{x}_i$ and $\boldsymbol{x}_k$ received are linearly independent at T_j. This allocation satisfies both upper bounds. The received signals at T_j yield:

$$\begin{aligned}\boldsymbol{y}_j = &\Big(\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ji}\\ \bar{\boldsymbol{u}}_{ji}\end{smallmatrix}\right] + \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{jk}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{jk}\\ \bar{\boldsymbol{u}}_{jk}\end{smallmatrix}\right]\Big) + \\ &\Big(\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ki}\ \bar{\boldsymbol{V}}_{ki}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ki}\\ \bar{\boldsymbol{u}}_{ki}\end{smallmatrix}\right] + \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{ik}\ \bar{\boldsymbol{V}}_{ik}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ik}\\ \bar{\boldsymbol{u}}_{ik}\end{smallmatrix}\right]\Big) + \boldsymbol{z}_j.\end{aligned}$$

The first sum in brackets describes the dedicated signals. The second sum in brackets describes the interfering signals at T_j (with $\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}$ and $\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}$ aligned). The signal and interference subspaces are linearly independent, since the composite $M_{\mathrm{Rx}}\times(3\tilde{d}+4\bar{d})$ matrix

$$[\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}\ \tilde{\boldsymbol{V}}_{ki}\ \bar{\boldsymbol{V}}_{ki}]\ \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{ik}]], \tag{5.172}$$

has full column rank, almost surely, due to:

$$3\tilde{d} + 4\bar{d} = 2M_{\mathrm{Tx}} - [2M_{\mathrm{Tx}} - M_{\mathrm{Rx}}]^+ < M_{\mathrm{Rx}}. \tag{5.173}$$

In the post-coding step, each receiver T_j uses a composite zero-forcing matrix $\boldsymbol{N}_j = [\boldsymbol{N}_{ji}^{\mathsf{T}} \boldsymbol{N}_{jk}^{\mathsf{T}}]^{\mathsf{T}}$ to separate and decode its two dedicated signals and to eliminate the interfering signals. The received signals $\boldsymbol{y}_j$ are filtered by the two corresponding zero-forcing matrices $\boldsymbol{N}_{ji}, \boldsymbol{N}_{jk} \in \mathbb{C}^{d \times M_{\mathrm{Rx}}}$, with:

$$\boldsymbol{N}_{ji}[\boldsymbol{H}_{jk}(\tilde{\boldsymbol{V}}_{jk}+\bar{\boldsymbol{V}}_{jk}+\tilde{\boldsymbol{V}}_{ik}+\bar{\boldsymbol{V}}_{ik})+\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ki}]=\boldsymbol{0}_{d\times(2d+\bar{d})}, \tag{5.174}$$

$$\boldsymbol{N}_{jk}[\boldsymbol{H}_{ji}(\tilde{\boldsymbol{V}}_{ji}+\bar{\boldsymbol{V}}_{ji}+\tilde{\boldsymbol{V}}_{ki}+\bar{\boldsymbol{V}}_{ki})+\boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{ik}]=\boldsymbol{0}_{d\times(2d+\bar{d})}, \tag{5.175}$$

so that filtering with $\boldsymbol{N}_{ji}\boldsymbol{y}_j$ and $\boldsymbol{N}_{jk}\boldsymbol{y}_j$ provides $d_{ji} + d_{jk} = 2d$ noisy interference-free streams of dedicated signals at T_j:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\left[\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji}\right] + \boldsymbol{N}_{ji}\boldsymbol{z}_j, \tag{5.176}$$

$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\left[\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk} + \bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk}\right] + \boldsymbol{N}_{jk}\boldsymbol{z}_j. \tag{5.177}$$

Thus, each user T_j can decode its two dedicated streams with:

$$d_{ji} + d_{jk} = 2d = 2(\tilde{d} + \bar{d}) = M_{\mathrm{Tx}}. \tag{5.178}$$

Altogether, $3M_{\mathrm{Tx}}$ DoF in the first term of (5.159) are achieved:

$$d_{\Sigma} \leq 6d = 3M_{\mathrm{Tx}}.$$

Case $M_{\mathrm{Tx}} \leq M_{\mathrm{Rx}}$ **with** $d_{\Sigma} = 2M_{\mathrm{Rx}}$

On the other hand, the dominant term in (5.159) yields $2M_{\mathrm{Rx}}$ if $2M_{\mathrm{Rx}} \leq 3M_{\mathrm{Tx}}$ holds. Then, $\mathsf{span}(\boldsymbol{H}_{ji}) \cap \mathsf{span}(\boldsymbol{H}_{jk})$ at T_j has $2M_{\mathrm{Tx}} - M_{\mathrm{Rx}} > \frac{1}{3}M_{\mathrm{Rx}}$ dimensions. We allocate:

$$d = \tilde{d} = \tfrac{1}{3}M_{\mathrm{Rx}}. \tag{5.179}$$

for all $i \in \mathcal{K}$. This allocation satisfies all upper bounds:

$$0 \leq 2d \leq M_{\mathrm{Tx}}, \tag{5.180}$$

$$0 \leq 3d \leq M_{\mathrm{Rx}}, \tag{5.181}$$

and no remaining dimensions are left at the receivers. The symbol streams $\tilde{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\tilde{d}\times 1}$ are pre-coded by the beam-forming matrices $\tilde{\boldsymbol{V}}_{ji} \in \mathbb{C}^{M_{\mathrm{Tx}}\times\tilde{d}}$ and are aligned analogously to (5.168). We have no symbol streams $\bar{\boldsymbol{u}}_{ji} \in \mathbb{C}^{\bar{d}\times 1}$, since $\bar{d} = 0$. Hence the received signal at T_j is:

$$\begin{aligned}\boldsymbol{y}_j = &\left(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk}\right) + \\ &\left(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}\tilde{\boldsymbol{u}}_{ki} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}\tilde{\boldsymbol{u}}_{ik}\right) + \boldsymbol{z}_j.\end{aligned} \tag{5.182}$$

The composite $M_{\mathrm{Rx}} \times 3\tilde{d}$-dimensional matrix:

$$[\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji}\ \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk}\ \boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}], \tag{5.183}$$

has full column rank, almost surely, so that dedicated and interfering signals are linearly independent.

For post-coding, the zero-forcing matrices are chosen as:

$$\boldsymbol{N}_{ji}(\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk} + \boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}) = \mathbf{0}_{\tilde{d}\times 2\tilde{d}}, \tag{5.184}$$
$$\boldsymbol{N}_{jk}(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}) = \mathbf{0}_{\tilde{d}\times 2\tilde{d}}, \tag{5.185}$$

such that the filtered signals yield:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \boldsymbol{N}_{ji}\boldsymbol{z}_j, \tag{5.186}$$
$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk} + \boldsymbol{N}_{jk}\boldsymbol{z}_j. \tag{5.187}$$

Each user T_j can decode noisy but interference-free versions of its two dedicated streams and achieves:

$$d_{ji} + d_{jk} = 2\tilde{d} = \tfrac{2}{3}M_{\mathrm{Rx}}, \tag{5.188}$$

so that the sum-DoF are:

$$d_{\Sigma} = 6d = 2M_{\mathrm{Rx}}. \tag{5.189}$$

Thence, the upper bound $\min(2M_{\mathrm{Rx}}, 3M_{\mathrm{Tx}})$ is shown to be achievable.

Case $M_{\mathrm{Tx}} > M_{\mathrm{Rx}}$ with $d_{\Sigma} = 2M_{\mathrm{Tx}}$

The upper bound (5.159) yields $2M_{\mathrm{Tx}}$ if $2M_{\mathrm{Tx}} \le 3M_{\mathrm{Rx}}$ holds. Again we use the symmetric DoF allocation as defined in (5.160) to (5.163). In this case, the following upper bounds must hold:

$$0 \le 2d \le M_{\mathrm{Rx}}, \tag{5.190}$$
$$0 \le 3d \le M_{\mathrm{Tx}}. \tag{5.191}$$

Since $M_{\mathrm{Tx}} > M_{\mathrm{Rx}}$, zero-forcing beam-forming [77, 78] is applicable and we allocate:

$$\bar{d} = M_{\mathrm{Tx}} - M_{\mathrm{Rx}} < \tfrac{1}{2}M_{\mathrm{Rx}} \tag{5.192}$$

dimensions. Analogous to Section 5.4.3, we pre-code the symbol-streams $\bar{\boldsymbol{u}}_{ji}$ and $\tilde{\boldsymbol{u}}_{ji}$ to construct the transmit signal $\boldsymbol{x}_i$ as in (5.166). The beam-forming matrix $\bar{\boldsymbol{V}}_{ki}$ has $M_{\mathrm{Tx}} \times \bar{d}$ dimensions and is designed to cast the interfering signal into the $(M_{\mathrm{Tx}} - M_{\mathrm{Rx}})$-dimensional null-space of T_j:

$$\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ki} = \mathbf{0}_{M_{\mathrm{Rx}}\times\bar{d}}. \tag{5.193}$$

For the next step, the number of remaining transmit and receive dimensions per user available for IA are:

$$\tilde{M}_{\mathrm{Tx}} = M_{\mathrm{Tx}} - 2(M_{\mathrm{Tx}} - M_{\mathrm{Rx}}) = 2M_{\mathrm{Rx}} - M_{\mathrm{Tx}}, \tag{5.194}$$
$$\tilde{M}_{\mathrm{Rx}} = M_{\mathrm{Rx}} - 2(M_{\mathrm{Tx}} - M_{\mathrm{Rx}}) = 3M_{\mathrm{Rx}} - 2M_{\mathrm{Tx}}. \tag{5.195}$$

Since $2\tilde{M}_{\mathrm{Tx}} > \tilde{M}_{\mathrm{Rx}}$ holds, the remaining dimensions suffice for IA. Furthermore, since $3\tilde{M}_{\mathrm{Tx}} > 2\tilde{M}_{\mathrm{Rx}}$, more than $\tilde{M}_{\mathrm{Rx}}/3$ dimensions are available for IA between each user pair (cf. Lemma 5.6). To establish a fair scheme, we set:

$$\tilde{d} = \tfrac{1}{3}\tilde{M}_{\mathrm{Rx}} = M_{\mathrm{Rx}} - \tfrac{2}{3}M_{\mathrm{Tx}}. \tag{5.196}$$

The beam-forming matrices $\tilde{\boldsymbol{V}}_{ki}$ and $\tilde{\boldsymbol{V}}_{ik}$, each with $M_{\mathrm{Tx}} \times \tilde{d}$ dimensions, are chosen such that the bidirectional interference signals are aligned at receiver T_j, as analogously done in (5.168). Due to zero-forcing beam-forming, the symbol streams $\bar{\boldsymbol{u}}_{ki}$ and $\bar{\boldsymbol{u}}_{ik}$ are not received at T_j, so that we obtain:

$$\begin{aligned}\boldsymbol{y}_j = &\left(\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\bar{\boldsymbol{V}}_{ji}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{ji}\\ \bar{\boldsymbol{u}}_{ji}\end{smallmatrix}\right] + \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\bar{\boldsymbol{V}}_{jk}]\left[\begin{smallmatrix}\tilde{\boldsymbol{u}}_{jk}\\ \bar{\boldsymbol{u}}_{jk}\end{smallmatrix}\right]\right) + \\ &\left(\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}\tilde{\boldsymbol{u}}_{ki} + \boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}\tilde{\boldsymbol{u}}_{ik}\right) + \boldsymbol{z}_j.\end{aligned} \tag{5.197}$$

The signal and interference subspaces are linearly independent, almost surely, since the composite matrix:

$$[\boldsymbol{H}_{ji}[\tilde{\boldsymbol{V}}_{ji}\ \bar{\boldsymbol{V}}_{ji}\ \tilde{\boldsymbol{V}}_{ki}]\ \boldsymbol{H}_{jk}[\tilde{\boldsymbol{V}}_{jk}\ \bar{\boldsymbol{V}}_{jk}]], \tag{5.198}$$

of $M_{\mathrm{Rx}} \times (3\tilde{d} + 2\bar{d})$ dimensions ($\boldsymbol{H}_{ji}\tilde{\boldsymbol{V}}_{ki}$ and $\boldsymbol{H}_{jk}\tilde{\boldsymbol{V}}_{ik}$ are aligned) has full column rank.

For post-coding at the receivers, we use zero-forcing matrices $\boldsymbol{N}_{ji}$ of $d \times M_{\mathrm{Rx}}$ dimensions as given in (5.174) and (5.175), but for differently allocated d according to (5.197). Analogously, the following signals are obtained after filtering:

$$\begin{aligned}\boldsymbol{N}_{ji}\boldsymbol{y}_j &= \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\left[\tilde{\boldsymbol{V}}_{ji}\tilde{\boldsymbol{u}}_{ji} + \bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji}\right] + \boldsymbol{N}_{ji}\boldsymbol{z}_j,\\ \boldsymbol{N}_{jk}\boldsymbol{y}_j &= \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\left[\tilde{\boldsymbol{V}}_{jk}\tilde{\boldsymbol{u}}_{jk} + \bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk}\right] + \boldsymbol{N}_{jk}\boldsymbol{z}_j.\end{aligned}$$

T_j decodes two noisy but interference-free dedicated streams:

$$d_{ji} + d_{jk} = 2d = 2(\tilde{d} + \bar{d}) = \tfrac{2}{3}M_{\mathrm{Tx}}, \tag{5.199}$$

so that the sum-DoF of $2M_{\mathrm{Tx}}$ are achieved:

$$d_\Sigma = 6d = 2M_{\mathrm{Tx}}. \tag{5.200}$$

Case $M_{\mathrm{Tx}} > M_{\mathrm{Rx}}$ with $d_\Sigma = 3M_{\mathrm{Rx}}$

In the case $2M_{\mathrm{Tx}} \geq 3M_{\mathrm{Rx}}$, the upper bound (5.159) yields $3M_{\mathrm{Rx}}$. Now it suffices to use zero-forcing beam-forming only. IA is actually not necessary for this case. We allocate the DoF:

$$d = \bar{d} = \tfrac{1}{2}M_{\mathrm{Rx}} \tag{5.201}$$

for all $k \in \mathcal{K}$, satisfying (5.190) and (5.191). We use the beam-forming matrices $\bar{\boldsymbol{V}}_{ji}$ with $M_{\mathrm{Tx}} \times \bar{d}$ dimensions and cast interference to the null-space of the undesired receivers:

$$\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ki} = \boldsymbol{0}_{M_{\mathrm{Rx}} \times \bar{d}}. \tag{5.202}$$

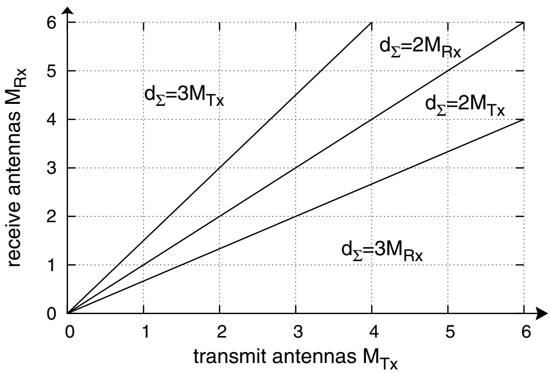


Figure 5.15: Sum-DoF d_Σ for the parameter plane of $0 \leq M_{\mathrm{Tx}} \leq 6$ transmit antennas and $0 \leq M_{\mathrm{Rx}} \leq 6$ receive antennas.

The received signal at receiver T_j is:

$$\boldsymbol{y}_j = \boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji} + \boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk} + \boldsymbol{z}_j. \tag{5.203}$$

The dedicated signals are linearly independent, almost surely, since the composite $M_{\mathrm{Rx}} \times 2\bar{d}$ matrix has full column rank:

$$[\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ji} \ \boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{jk}]. \tag{5.204}$$

For post-coding at receiver T_j, we use the zero-forcing matrices $\boldsymbol{N}_{ji}$ and $\boldsymbol{N}_{jk}$ of $\bar{d} \times M_{\mathrm{Rx}}$ dimensions each so that (5.184) and (5.185) hold. We obtain the following filtered signals:

$$\boldsymbol{N}_{ji}\boldsymbol{y}_j = \boldsymbol{N}_{ji}\boldsymbol{H}_{ji}\bar{\boldsymbol{V}}_{ji}\bar{\boldsymbol{u}}_{ji} + \boldsymbol{N}_{ji}\boldsymbol{z}_j, \tag{5.205}$$

$$\boldsymbol{N}_{jk}\boldsymbol{y}_j = \boldsymbol{N}_{jk}\boldsymbol{H}_{jk}\bar{\boldsymbol{V}}_{jk}\bar{\boldsymbol{u}}_{jk} + \boldsymbol{N}_{jk}\boldsymbol{z}_j. \tag{5.206}$$

Each receiver T_j can decode:

$$d_{ji} + d_{jk} = 2d = 2\bar{d} = M_{\mathrm{Rx}}, \tag{5.207}$$

and achieves the sum-DoF of:

$$d_\Sigma = 6d = 3M_{\mathrm{Rx}}. \tag{5.208}$$

Altogether, the upper bound $\min(2M_{\mathrm{Tx}}, 3M_{\mathrm{Rx}})$ is also shown to be achievable. Note that complete fairness is maintained in each case. This concludes the proof of Theorem 5.8.

5.4.4 Discussion

The parameter plane of the symmetric DoF depicted in Figure 5.15 provides a symmetry along the intersecting line $M_{\mathrm{T}} = M_{\mathrm{R}}$ for all parameters M_{T} and M_{R}. At that line, the antenna parameters of the achieved DoF are swapped since null-space beam-forming and linear independent beam-forming are swapped.

5.5 Extending the CPCM by Multi-Antenna Constraints

In order to expose further analogies between the CPCM and the GMCM, additional constraints are imposed to the CPCM of the 3-way channel, for distinct $i, j, k \in \mathcal{K}$:

$$m_{ji} \leq \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}), \tag{5.209}$$

$$m_{ji} + m_{ki} \leq M_{\mathrm{Tx}}, \tag{5.210}$$

$$m_{ij} + m_{ik} \leq M_{\mathrm{Rx}}. \tag{5.211}$$

These inequalities are intended to mimic the MIMO antenna constraints for a number of M_{Tx} transmit antennas and M_{Rx} receive antennas at each user T_i. We now investigate the intersection subspace of two transmitted polynomials within a received polynomial (cf. Lemma 5.6) in terms of the CPCM.

Let $q_i(x)$ denote a polynomial of M_{Rx} dimensions for each transmitter Tx_i, $i \in \{1, 2\}$:

$$q_i(x) = \sum_{k=0}^{M_{\mathrm{Rx}}-1} q_i^{[k]} x^k. \tag{5.212}$$

The two polynomials $q_1(x)$ and $q_2(x)$ from the transceivers Tx_1 and Tx_2 have M_{Rx} dimensions, since the intersection space of all dedicated signals must be decodable within the received signal space of a receiver Rx. We use binary coefficients $q_i^{[0]}, \ldots, q_i^{[M_{\mathrm{Rx}}-1]} \in \{0, 1\}$, to simplify the computation of the number of intersecting dimensions at Rx.

At each Tx_i, a number of only $\min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})$ coefficients in $q_i^{[\cdot]}$ is set to value one at random positions, to indicate that these dimensions are occupied by information. The remaining $[M_{\mathrm{Rx}} - \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})]^+$ unused coefficients are set to zero at each Tx_i.

Next, we compute the intersection space of the non-zero coefficients from the shifted polynomials $q_1(x)$ and $q_2(x)$ as received at Rx. The received polynomial $q(x)$ has also M_{Rx} dimensions and is expressed by:

$$q(x) = d_{11} q_1(x) \wedge d_{21} q_2(x) \tag{5.213}$$

$$= \sum_{k=0}^{M_{\mathrm{Rx}}-1} q^{[k]} x^k, \tag{5.214}$$

with the bit-wise and-operation (symbol: $\wedge$). The sought intersection space corresponds to the number of non-zero coefficients in $q(x)$ which is simply computed by the accumulative sum of the received coefficients $q^{[k]} \in \{0, 1\}$:

$$q(1) = \sum_{k=0}^{M_{\mathrm{Rx}}-1} q^{[k]}.$$

Lemma 5.9. *The intersection space of two M_{Rx}-dimensional interfering polynomials $q_1(x)$ and $q_2(x)$ of $q_i(1) = \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})$ non-zero coefficients, with $i \in \{1, 2\}$, is bounded by:*

$$0 \leq [2\min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}) - M_{\mathrm{Rx}}]^+ \leq q(1) \leq \min(M_{\mathrm{Rx}}, M_{\mathrm{Tx}}) \tag{5.215}$$

non-zero coefficients in the M_{Rx}-dimensional received polynomial $q(x)$.

Proof:

(a) *Upper bound:* $\min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}) \geq q(1)$ *dimensions:*
The maximal number of non-zero coefficients in the intersection $d_{11}q_1(x) \wedge d_{21}q_2(x)$ is $q(1) = \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})$ and it is achieved if the non-zero coefficients of both shifted polynomials $d_{11}q_1(x)$ and $d_{12}q_2(x)$, each with $q_i(x) = \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})$ non-zero coefficients, fully overlap at Rx. Then, it holds $q_1(x) = q_2(x) = q(x)$, and we obtain $q(1) = \min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}})$.

(b) *Lower bound:* $0 \leq [2\min(M_{\mathrm{Rx}}, M_{\mathrm{Tx}}) - M_{\mathrm{Rx}}]^+ \leq q(1)$ *dimensions:*
At least $\min(M_{\mathrm{Rx}}, M_{\mathrm{Tx}})$ coefficients are non-zero in both polynomials $q_1(x)$ and $q_2(x)$. Let $M_{\mathrm{Tx}} < M_{\mathrm{Rx}}$, so that $\min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}) = M_{\mathrm{Tx}} < \max(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}) = M_{\mathrm{Rx}}$. Let Tx_1 fix its M_{Tx} non-zero coefficients first. This leaves $(M_{\mathrm{Rx}} - M_{\mathrm{Tx}})^+$ dimensions available for M_{Tx} non-zero coefficients in $q(x)$. In case $M_{\mathrm{Rx}} \geq 2M_{\mathrm{Tx}}$ holds, it is possible that all M_{Tx} non-zero coefficients are distinct from the M_{Tx} already fixed coefficients at Tx_1, yielding $q(1) = 0$. Otherwise, if $M_{\mathrm{Rx}} \leq 2M_{\mathrm{Tx}}$ holds, there will always be an intersection of non-zero coefficients in at least $[2\min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}) - M_{\mathrm{Rx}}]^+$ dimensions. Thus, we obtain the lower bound $[2\min(M_{\mathrm{Tx}}, M_{\mathrm{Rx}}) - M_{\mathrm{Rx}}]^+$.

As a result, the present lemma is proven. ■

Note that the lower bound provided here exactly coincides with the number of dimensions of the intersection subspaces as in Lemma 5.6 for $M_{\mathrm{Tx}_i} = M_{\mathrm{Tx}}$. By imposing these antenna constraints, we established an interesting analogy in the CPCM and the GMCM regarding the intersection space of two transmitted signals.

5.6 Summary

In this chapter, we have investigated the 3-way channel and its relationship to the 3-user Y-channel. We have observed that the Degrees-of-Freedom of the two aforementioned channels coincide when considering the cyclic polynomial channel model. Our main result was that these channels were interlinked via a Δ-Y relationship, which was inspired by the well-known transformation in the circuit theory of resistor networks. Furthermore, in contrast to most of the other previously considered channels, these two channels were not subject to further constraints on the channel matrices. In order to gain from these insights as given by our proposed channel model, we have examined the 3-way channel w. r. t. two other well-established channel models: the linear deterministic channel model and the Gaussian MIMO channel model with constant coefficients.

Within the linear deterministic channel model, the Δ-Y relationship was also extended to characterize the capacity region of the linear deterministic 3-way channel with reciprocal channel coefficients in the uplink and downlink. We have investigated a capacity-achieving signal alignment scheme that has performed two-way relaying over a single user, and also a capacity-achieving interference alignment scheme for a symmetric special case.

Then, we have considered two cases of the MIMO 3-way channel: The $M_i \times M_i$ 3-way channel and the $M_{\mathrm{Tx}} \times M_{\mathrm{Rx}}$ 3-way channel. Due to the absence of further constraints on the channel matrices, the problem of common eigenvectors in separate subspaces has disappeared for the related MIMO channels. Thus, we have discussed a 3-way channel with M_i transmit and receive antennas at each user. We have provided a scheme that achieved the upper bound of $2M_2$ Degrees-of-Freedom. Basically, it has already sufficed to ignore messages between on the weakest subchannel. But in order to maintain fairness between the users, we have employed a MIMO IA and zero-forcing scheme to provide a fair rate-allocation, that has achieved the sum-Degrees-of-Freedom of $2M_2$. In the second case, we have discussed a 3-way channel with symmetrically equipped users having M_{Tx} transmit and M_{Rx} receive antennas at each transceiver. Accordingly, we have provided IA schemes that have achieve the upper bounds on the symmetric Degrees-of-Freedom.

Altogether, this chapter has provided a comparison between the three different channel models of interest for the example of the 3-way channel.

6 Exploring Practical Applications for Cyclic Interference Alignment

Up to this point, we discussed optimal communication strategies for several multi-user channels described by the CPCM. Now, we turn our attention to some practical issues of IA and cyclic IA by propagation delay on the CPCM in particular:

Potential practical applications of cyclic IA, and

Node placement in Euclidean space for cyclic IA by propagation delay[1].

6.1 On Practical Implementations of IA

The high rate-gains theoretically achievable by IA as introduced in [3] allured numerous researchers in the past few years. There are plenty of works dealing with IA under idealized assumptions as those ones cited in the survey [12]. However, embedding the theoretical concepts of IA into practical systems remains a very challenging task [92]. The restrictions and demands of IA are manifold. The main requirements on IA predominantly discussed in the literature [12], [92] are briefly summarized here:

Approximating the channel capacity in terms of the DoF becomes accurate at high SNR only. However, a practical implementation of mobile devices providing high SNR communication is limited.

The necessity to obtain exact global CSI[2] for each user is a computationally extensive task [40] and increases exponentially with K users. Furthermore, a severe overhead is involved for the acquisition of CSI.

IA schemes are very fragile if there are imperfections in the acquired CSI, as shown in [94], [92] and [95] for instance. If the accuracy drops below a certain threshold, it leads to detrimental leakage interference [40] and thus to an immense performance loss at each receiver.

Most IA schemes demand full and highly exact synchronization (in time and frequency) among multiple users, which is very challenging in practical implementations.

Computing obtain optimal pre-coding matrices is extensive [95]. Algorithms converge rather slowly and residual imperfect alignment causes leakage interference.

[1] Parts of this work have been published in [27].
[2] However, this does not concern *blind IA* [93].

The infinite time-extensions [3] in time-varying MIMO channels, and the pairing of complementary channels as in ergodic IA [15], cause exponential decoding delays.

Depending on the specific underlying applied IA scheme, there are yet many other fragile requirements to be considered that are not mentioned in the list above. Nonetheless, a few works already focus on the initial implementation of a realistic practical testbed for MIMO IA. For a first feasibility check, the work [96] uses previously measured channel coefficients of a real environment while the actual IA scheme works in an offline simulation. It shows that the basic assumptions on the channel coefficients in the theoretical works are indeed feasible. Among other works, [61], [95], [97], [98], and [99] discuss the implementations of various IA testbeds. The results are promising, but they do not yet achieve the full potential proposed by theory. This weakness is due to the limitations and extensive demands enlisted above.

An important aspect to note here is that not all multi-user scenarios must necessarily involve IA. Depending on the respective interference power at each user, a proper combination of treating interference as noise, cancelling strong interference, and aligning only particular interference signals is yet open to be formulated. We conjecture that the optimal strategies will involve classifying specific interference scenarios in order to determine to what extent the application of IA is beneficial. In the following part, we focus on the requirements for a potential application of cyclic IA by propagation delay.

6.2 Cyclic IA by Propagation Delay

If the cyclic shifts as imposed by the channel matrix are unrolled over time, we may interpret the received signals undergoing a discrete propagation delay. For a proper application of cyclic IA, the decoders must take a transient settling time for the initial phase of the communication into account. But this initial transient period is only limited by the subchannel with the longest propagation delay. Afterwards, a stationary state follows that shows the same behaviour as the CPCM with n dimensions. If the transmitters are switched off at the end of the communication, a final transient period must be taken into account, as well. But these transient periods are negligible for permanent communication. With this interpretation, we the discrete propagation delay-based examples provided in [3, Appendix I], [37, Sect. 1] are generalized on the one hand, and on the other hand, the discussions of [51] and [16] are extended. There are some beneficial properties, but also a few challenging requirements, that must be taken into account when considering IA by propagation delay:

Advantages: It suffices to use very simple user-nodes with a single antenna, for omnidirectional signal propagation (i. e., no beam-forming). Only the CSI of the dedicated link must be known at the intended receivers. Although, the exact knowledge of all propagation delays between each user is obligatory, it can be measured simpler than the CSI (e. g., using a synchronized atomic clock, as in the global positioning system (GPS)). Minor variations in the propagation delay, i. e., jitter, can be compensated by using guard intervals if the frame durations are sufficiently long. For long frame durations, we can assume that the conveyed messages may also be protected by a conventional inner code to counteract noise.

Disadvantages: The propagation delays must be fairly static. Especially synchronization errors and multi-path propagation will impose leakage interference. For multiple users, exceeding $K \geq 3$, the total number of feasible channel delay matrices reduces quickly due to the separability conditions. This detrimental effect uncovers an interesting parallel to the *bandwidth-scaling* property discussed for IA by propagation delay with LoS paths in [51]. In other words, the bandwidth must scale sufficiently with the number of users K to provide a sufficiently large space to accommodate the dedicated signals. This property is actually conflicting with the fundamental demand of IA to include many users. However, note that in many of the different IA schemes proposed in the literature [12], comparable conflicting effects occur. It is clear that propagation delays are not discrete, but real-valued of course. However, this does not impose a major problem, since scaling-up the message demands in $\boldsymbol{M}$, inherently scales with period-length n, so that real-valued delays in $\boldsymbol{D}$ can be approximated with sufficient accuracy.

Two communication scenarios, that seem to fit for (cyclic) IA by propagation delay at best, are: *Deep-space* communications and *underwater* communications. Both these setups may provide very specific delay and multi-path properties that are advantageous to implement cyclic IA by propagation delay.

6.2.1 Deep-Space Communications

Signals in a wireless deep-space scenario travel at the speed of light with $c_{\mathrm{L}} \approx 3{\cdot}10^8 m/s$. This scenario naturally scales with very long propagation delays and LoS signalling paths without multi-path scattering. For a distance of multiples of 300m between each transmitter and receiver, a sufficiently short frame-duration would be $T_\Delta \approx 1\mu s$ as proposed in [16]. This implies quite a high frequency of $f_\Delta = T_\Delta^{-1} = 10\text{MHz}$, and even higher carrier frequencies for multiple symbols within each frame. Since multiple communicating users are naturally distributed at different locations, each user-pair will experience different propagation delays. In the work [100], schemes for IA by propagation delay are discussed for a K-user interference channel. The given schemes are modelled for a corresponding satellite system with real-valued propagation delays.

6.2.2 Underwater Communications

As propagation delay[3] is also a significant property observed in underwater communications with acoustic signals, another interesting opportunity for cyclic IA by propagation delay is available in [102]. From a simplified perspective, we may assume that acoustic signals travel at the speed of sound in the medium of seawater with $c_{\mathrm{S}} \approx 1.5{\cdot}10^3$ m/s. For a distance of multiples of 300 m between each transmitter and receiver, a sufficiently short frame duration would be $T_\Delta \approx 200$ ms, respectively. In this light, slight synchronization inaccuracies will be less degrading to the communication scheme than the previous case.

However, a more accurate and realistic acoustic underwater channel model undergoes some other severe detrimental effects as discussed in [103]. Especially potential multi-

[3]Propagation delay is sometimes called *latency* in oceanic communications engineering (e. g., [101]).

path effects caused by reflections on the surface and at the ocean floor and attenuation might weaken the LoS communication link. Furthermore, high SNR transmissions are an impractical option, since the available energy for the mobile transmitters (e. g., ships, submarines, buoys, aquanauts, sensors) is usually limited.

6.2.3 Node Placement in Euclidean Space

Since the number of feasible channel matrices is restrictive, a contrary approach is to design the propagation delays a wireless communication channel itself. *Node placement* terms a procedure of how to place user-nodes in Euclidean space enabling the discrete delays for (Cyclic) IA by propagation delay [16]. For the sake of simplicity, we assume that propagation delay is proportional to the Euclidean distance between each user. We neglect further wireless effects as, e. g., multi-path propagation, path loss and fading and consider a static LoS environment. These particular assumptions are also used in [16], [104], [105], [102] and [100] for the derivation of node placement schemes in multi-user interference channels.

The Euclidean distances between each user are denoted by a symmetric *dissimilarity matrix* $\boldsymbol{\Delta} = (\delta_{ji})_{1\leq j,i\leq 4}$ with the entries $\delta_{ji} \in \mathbb{N}_{>0}$ and a zero diagonal [101]. We normalize the relationship between the propagation delay d_{ji} and the Euclidean distance δ_{ji} by $d_{ji} = x^{\delta_{ji}}$. The dissimilarity matrix $\boldsymbol{\Delta}$ of a communication channel with transmitters and receivers can be decomposed into four blocks:

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C}^\top \\ \boldsymbol{C} & \boldsymbol{A} \end{pmatrix}, \tag{6.1}$$

with fixed entries in the matrix $\boldsymbol{C}$ and variable entries $a, b \in \mathbb{R}^+$ in $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. The elements of the submatrix $\boldsymbol{A}$ correspond to distances between the transmitters and the elements of $\boldsymbol{B}$ to distances between the receivers. The elements of the submatrix $\boldsymbol{C}$ correspond to the fixed delay offset exponents of the channel matrix $\boldsymbol{D}$ from the CPCM.

The problem of node placement is analogous to an *Euclidean embedding* of the dissimilarity matrix $\boldsymbol{\Delta}$ according to [104] and [105]. To derive the sought distances b and a, the receivers Rx_j and k, $j \neq k \in \{3, 4\}$, must be positioned such that all the given distances in $\boldsymbol{C}$ are fulfilled.

The objective of Euclidean embedding is to find a solution of *placement vectors* $\boldsymbol{x}_i \in \mathbb{R}^m$ for all users $i = 1, \ldots, 4$ within the lowest number of Euclidean dimensions m yet satisfying the fixed distances in $\boldsymbol{\Delta}$. In our case it is desirable to find a practically most relevant solution in only $m \leq 3$ dimensions for a spatial arrangement of users.

6.2.4 2-User X-Channel

A sufficiently simple example to describe node placement is the 2-user X-channel, since there are only four users and only minimal constraints to the channel matrix to be considered. Recall that this particular channel has been discussed in Section 3.3 in terms of the CPCM for general message lengths. For notational reasons, we define the following indexing for the 4 users $\mathrm{Tx}_1, \mathrm{Tx}_2, \mathrm{Rx}_1$ and Rx_2 in the X-channel:

$$\mathrm{Tx}_1 \mathrel{\hat{=}} 1,\ \mathrm{Tx}_2 \mathrel{\hat{=}} 2,\ \mathrm{Rx}_1 \mathrel{\hat{=}} 3,\ \mathrm{Rx}_2 \mathrel{\hat{=}} 4. \tag{6.2}$$

This change of indices will also be used for the delay exponents δ_{ji}. Note that the condition $\mathsf{det}(\boldsymbol{D}) \not\equiv 0 \bmod (x^3 - 1)$ from Theorem 5.7 leads to an equivalent condition for matrix $\boldsymbol{C}$:

$$\delta_{31} + \delta_{42} \not\equiv \delta_{41} + \delta_{32} \pmod 3. \tag{6.3}$$

Figure 6.1 depicts an exemplary solution of node placement for cyclic IA by delay

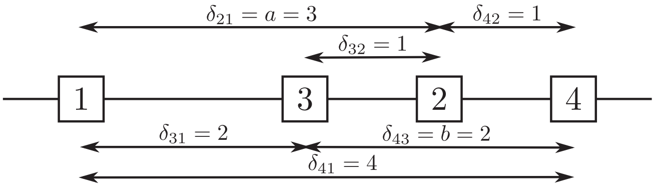


Figure 6.1: A one-dimensional solution for the node placement of a given matrix $\boldsymbol{C}$ with $\delta_{31} = 2$, $\delta_{32} = 1$, $\delta_{41} = 4$ and $\delta_{42} = 1$ is shown. The numbered boxes indicate the users. The parameters of the dissimilarity matrix $\boldsymbol{\Delta}$ are $b = 2$ and $a = 3$. Cyclic IA is possible since condition (6.3) is fulfilled.

on the X-channel in only one dimension. Another exemplary node placement in two dimensions is depicted in Figure 6.2.

We may set the variables $\delta_{12} = \delta_{21} = b \in \mathbb{R}^+$ and $\delta_{34} = \delta_{43} = a \in \mathbb{R}^+$ due to symmetry and $\delta_{ii} = 0$, for $i = 1, ..., 4$, and we obtain the sought submatrices:

$$\boldsymbol{B} = b(\mathbf{1}_{2\times 2} - \boldsymbol{I}_{2\times 2}), \tag{6.4}$$
$$\boldsymbol{A} = a(\mathbf{1}_{2\times 2} - \boldsymbol{I}_{2\times 2}). \tag{6.5}$$

There is a feasible solution in two dimensions, if both transmitters satisfy all of the following (four) triangle inequalities with $j, k \in \{3, 4\}$ and $i, l \in \{1, 2\}$:

$$0 < \|\delta_{ji} - \delta_{jl}\|_2 \le b \le \|\delta_{ji}\|_2 + \|\delta_{jl}\|_2, \tag{6.6}$$
$$0 < \|\delta_{ji} - \delta_{ki}\|_2 \le a \le \|\delta_{ji}\|_2 + \|\delta_{ki}\|_2. \tag{6.7}$$

These inequalities must be non-zero because the users may not overlap at the same point in the Euclidean space. If the lower and upper bounds support a solution with $b, a \in \mathbb{R}$, then a 2-dimensional node placement exists.

The solution is easily derived from elementary geometry: W. l. o. g. we can position Tx_i on a reference point $(0, 0)$. To position Tx_l, $i \ne l$, some b satisfying condition (6.6) can be fixed on a straight line that originates in the reference point $(0, 0)$. We may fix Tx_l at $(b, 0)$ for instance. Then the valid positions for receivers Rx_j and Rx_k, $j \ne k$, can be determined, as well. Valid positions for Rx_j are the intersecting points of the two circles $\mathcal{O}_{ji}$ and $\mathcal{O}_{jl}$ with $j \ne k \in \{3, 4\}$, $i \ne l \in \{1, 2\}$:

$$\begin{aligned} \mathcal{O}_{ji}: &\quad x^2 + y^2 &= \delta_{ji}^2, \\ \mathcal{O}_{jl}: &\quad (x - b)^2 + y^2 &= \delta_{jl}^2. \end{aligned} \tag{6.8}$$

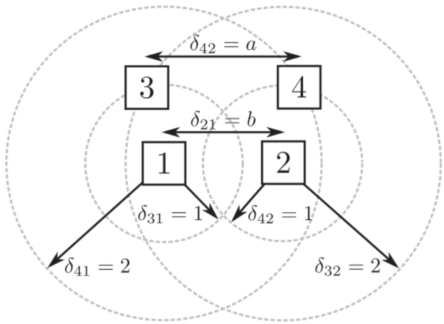


Figure 6.2: A two-dimensional node placement solution is shown for a given matrix $\boldsymbol{C}$ with $\delta_{31} = 1$, $\delta_{32} = 2$, $\delta_{41} = 2$ and $\delta_{42} = 1$. The condition (6.3) is fulfilled, parameter b satisfies (6.6) and a satisfies (6.7).

The valid positions for Rx_k yield from the intersection points of the circles:

$$\begin{aligned} \mathcal{O}_{ki}: \quad x^2 + y^2 &= \delta_{ki}^2, \\ \mathcal{O}_{kl}: \quad (x-b)^2 + y^2 &= \delta_{kl}^2. \end{aligned} \tag{6.9}$$

accordingly. Condition (6.7) is satisfied by construction, due to the fact that the circles $\mathcal{O}_{ji}$, $\mathcal{O}_{ki}$ are concentric around node i and the circles $\mathcal{O}_{jl}$, $\mathcal{O}_{kl}$ concentric around node l.

The resulting placement vectors are computed as:

$$\begin{aligned} \boldsymbol{x}_i &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \boldsymbol{x}_l &= \begin{pmatrix} b \\ 0 \end{pmatrix}, \\ \boldsymbol{x}_j &= \begin{pmatrix} \frac{\delta_{ji}^2 - \delta_{jl}^2 + b^2}{2b} \\ \pm\sqrt{\alpha_j} \end{pmatrix}, & \boldsymbol{x}_k &= \begin{pmatrix} \frac{\delta_{ki}^2 - \delta_{kl}^2 + b^2}{2b} \\ \pm\sqrt{\alpha_k} \end{pmatrix}, \end{aligned} \tag{6.10}$$

with the indices $j \neq k \in \{3, 4\}$ and $i \neq l \in \{1, 2\}$, and the discriminants:

$$\alpha_j = \delta_{ji}^2 - \frac{\delta_{ji}^2 - \delta_{jl}^2 + b^2}{2b}, \tag{6.11}$$

$$\alpha_k = \delta_{ki}^2 - \frac{\delta_{ki}^2 - \delta_{kl}^2 + b^2}{2b}. \tag{6.12}$$

If both discriminants α_j, α_k are greater than zero, then a two-dimensional solution exists. If $\alpha_3 = \alpha_4 = 0$, the solution is one-dimensional. A negative discriminant states that there is no feasible solution for node placement. The remaining Euclidean distance a between the receivers Rx_j and Rx_k is computed by:

$$a = \|\boldsymbol{x}_j - \boldsymbol{x}_k\|_2. \tag{6.13}$$

with $j \neq k \in \{3, 4\}$. Moreover, such a positioning of users is invariant to rotation and translation w. r. t. a reference point.

An analogous dual solution is to fix receiver Rx_j at point $(0,0)$ and Rx_k at point $\boldsymbol{x}_k = (a,0)^\mathsf{T}$ satisfying condition (6.7) and finding the intersection points of the corresponding two circle pairs around Rx_j and Rx_k.

The extension to $m = 3$ dimensions would include an additional z-coordinate and the computation of two intersection circles for the four intersecting spheres. The 3-dimensional solution is also a rotational body around the connecting line δ_{li} between Tx_i and Tx_l of the 2-dimensional solution. Again, it is invariant to rotation and translation w. r. t. a fixed reference point.

6.3 Summary

In this chapter, we have given a brief overview of the practical challenges of the predominant interference alignment schemes in general. Next, we have discussed the opportunities and challenges of a potential practical application of *cyclic interference alignment by propagation delay*. We have observe that including the effect of long propagation delays in multi-user communication is indeed practically relevant, but quite limited so far. In the current stage, a fully practical application of cyclic IA by propagation delay has not been implemented yet. However, there are actually some very specific scenarios in deep-space and underwater communication systems that consider IA by propagation delay.

In the literature, the investigation of interference alignment by propagation delay is tightly connected to the problem of *node placement*, i. e., the positioning of transmitters and receivers in Euclidean space enabling interference alignment. In this light, have we investigated the node placement problem for the 2-user X-channel and we have provided conditions on its feasibility.

7 Conclusions

In this thesis, a novel channel model has been developed on the basis of a polynomial ring representation - the *cyclic polynomial channel model* (CPCM). The proposed model was mainly inspired by *interference alignment by propagation delay*, the algebraically convenient description of *cyclic codes*, and by the *linear deterministic channel model* (LDCM) as introduced in the seminal work of Avestimehr et al. The CPCM described multi-user communication systems with interference. Therein, each transmitted signal experienced an individual cyclic shift and at the receivers, those shifted signals interfered with each other.

Several different multi-user communication systems have been investigated from an information-theoretic perspective applying the CPCM. These systems ranged from elementary unidirectional multi-user channels to multi-way and multi-hop networks with relays. We have derived optimal communication strategies to achieve maximal datarates. These schemes involved a cyclic polynomial representation of *interference alignment*, *signal alignment*, *interference neutralization*, *orthogonal multiple-access* and *linear coding schemes* in particular. It has turned out that utilizing the CPCM is convenient for an approximation of optimal communication schemes when compared to the conventional *Gaussian MIMO channel model* (GMCM) and the LDCM. We have defined a set of *separability conditions* to allow for linear decodability at the receivers. This step has assisted us in classifying the intricate interference patterns of multiple users and it has facilitated the derivation of feasibility and optimality of our proposed communication schemes. As a result, even for channels with arbitrary asymmetric cyclic shifts, it has been shown that optimal solutions can be represented in a compact form.

Besides capacity characterization in terms of the CPCM, some interesting phenomena hidden within the discussed interference networks have been discovered. In particular, we have elaborated a *complementary reciprocal symmetry of alignment* that served as a key property for the capacity-achieving scheme of the 2-user X-network. It described how pairs of aligned signal and interference patterns at different receivers are related. Furthermore, we have described a duality between cyclic interference alignment and cyclic interference neutralization in a 3-user X-network with a minimal number of cognitive messages. Motivated by the well-known transformation in circuit theory, a *Δ-Y duality relationship* has been formulated for 3-way channels and 3-user Y-channels.

To further substantiate our proposed model, we have generalized the insights gained from the CPCM to design optimal schemes for the 3-way channel described by the LDCM and the GMCM. In both cases, upper bounds and achievable schemes have been derived w.r.t. the given model. In our last step, we have briefly explored potential applications of cyclic interference alignment in networks with long propagation delays, e. g., as in deep-space and underwater communication systems. Moreover, we have discussed a node placement scheme in Euclidean space for the 2-user X- channel.

In summary, this thesis has provided an elaborate theoretical analysis and optimal coding schemes for various multi-user interference networks modelled with arbitrary cyclic shifted communication links.

7.1 Outlook

There are still plenty of interesting interference network problems open to be considered using the CPCM.

In particular, we intend to generalize both the 3 -way channel and the 3 -user Y- channel, as discussed in Chapter 5, to K-users with $K \geq 3$. A solution would also answer the open question whether a *generalization of the* Δ-Y *relationship* exists for K users. Then, the corresponding results in the CPCM might facilitate the derivations for schemes on the more conventional GMCM and the LDCM. During our studies, we also encountered the concept of *interference forwarding (IFWD)*, i. e., (aligned) interference signals are intentionally forwarded by a relay to provide side-information to the destinations, so that cancellation can be performed, as discussed in [106]. We already observed in some examples that IFWD is also useful for the achievable scheme of the 3 -way channel and the multi-user extensions. This observation might eventually lead to some interesting *duality relationships between cyclic interference alignment, interference neutralization and interference forwarding.*

Furthermore, as perfect cyclic interference alignment schemes severely limit the ratio of feasible channel matrices for the K-user interference channel, we intend to study its *generalization to arbitrary message lengths* and investigate to what extend the conditions can be relaxed. In a subsequent step, we intend to permit imperfect interference alignment that will lead to limited leakage-interference. This approach can be used to characterize a *trade-off for the achievable data-rate versus the channel-feasibility ratio.*

There is also an opportunity to extend the underlying framework of the proposed channel model itself. A first step would be to use *channel matrices with sparse polynomials* instead of the matrices with monomials only. Such an approach would support modelling the interaction of a finite number of delayed multi-path signal echos, since they are of major concern for cyclic interference alignment by propagation delay. Some preliminary studies have already shown that, despite these echos, decoding schemes can be designed over the time-unrolled CPCM.

Appendix

A Constraints of the 3-User X-Network with Minimal Backhaul

In the following, we carefully prove that the IAC scheme in (3.146) to (3.153) satisfies all separability conditions (3.126) to (3.139) leading to the conditions of Theorem 3.12. The two particular exceptions given by (3.150) and (3.153) are neglected. The derivations are enlisted column-wise and congruences are taken modulo $x^5 - 1$.

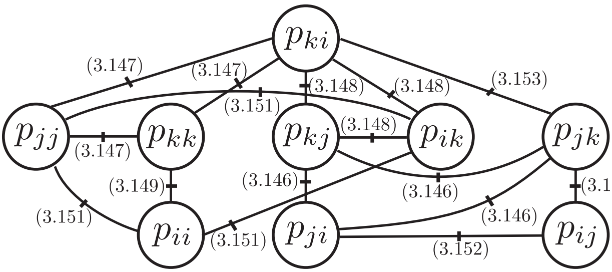


Figure A.1: Adjacency graph of the cyclic IA scheme in (3.146) to (3.153). Solid lines indicate the assignments for a fixed p_{ki} given in the proof of Theorem 3.12, and dashed lines the remaining conditions that must be checked for feasibility.

$$
\begin{aligned}
(3.126): x^{p_{ki}} &\not\equiv x^{p_{ji}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ki}} &\not\equiv d_{ij} d_{ii}^{-1} x^{p_{kj}} \\
\overset{(3.148)}{\Rightarrow} x^{p_{ki}} &\not\equiv d_{ij} d_{ii}^{-1} d_{ji} d_{jj}^{-1} x^{p_{ki}} \\
\Rightarrow 0 &\not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \Rightarrow \text{(iv)}
\end{aligned}
$$

$$
\begin{aligned}
(3.127): x^{p_{ji}} &\not\equiv x^{p_{ii}} \\
\overset{(3.151)}{\Rightarrow} x^{p_{ji}} &\not\equiv d_{kk} d_{ki}^{-1} x^{p_{ik}} \\
\overset{(3.148)}{\Rightarrow} x^{p_{ji}} &\not\equiv d_{kk} d_{ki}^{-1} d_{jj} d_{jk}^{-1} x^{p_{kj}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ji}} &\not\equiv d_{kk} d_{ki}^{-1} d_{jj} d_{jk}^{-1} d_{ii} d_{ij}^{-1} x^{p_{ji}} \\
\Rightarrow d_{ii} d_{jj} d_{kk} &\not\equiv d_{ij} d_{jk} d_{ki} \Rightarrow \text{(ix)}
\end{aligned}
$$

$$
\begin{aligned}
(3.128): x^{p_{ki}} &\not\equiv x^{p_{ii}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ki}} &\not\equiv d_{jk} d_{ji}^{-1} x^{p_{kk}} \\
\overset{(3.148)}{\Rightarrow} x^{p_{ki}} &\not\equiv d_{jk} d_{ji}^{-1} d_{ii} d_{ik}^{-1} x^{p_{ki}} \\
\Rightarrow 0 &\not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.129): d_{ij} x^{p_{ij}} &\not\equiv d_{ik} x^{p_{ik}} \\
\overset{(3.148)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ik} d_{ij}^{-1} d_{jj} d_{jk}^{-1} x^{p_{kj}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ik} d_{ij}^{-1} d_{jj} d_{jk}^{-1} d_{ii} d_{ij}^{-1} x^{p_{ji}} \\
\overset{(3.152)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ik} d_{ij}^{-1} d_{jj} d_{jk}^{-1} \\
&\qquad d_{ii} d_{ij}^{-1} d_{kj} d_{ki}^{-1} x^{p_{ij}} \\
\overset{(i)}{\Rightarrow} d_{ii} d_{jj} d_{ik} d_{kj} &\not\equiv d_{ji} d_{kj} d_{ik} d_{ij} \\
\Rightarrow 0 &\not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \Rightarrow \text{(iv)}
\end{aligned}
$$

$$
\begin{aligned}
(3.130): d_{ii}x^{p_{ii}} &\not\equiv d_{ij}x^{p_{ij}} \\
\overset{(3.150)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-1}d_{jk}d_{jj}^{-1}x^{p_{jk}} \\
\overset{(3.153)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-1}d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{ki}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-1}d_{jk}d_{jj}^{-1} \\
&\quad d_{ki}d_{kk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}} \\
\overset{(3.149)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-2}d_{jk}d_{jj}^{-1}d_{ki} \\
&\quad d_{kk}^{-1}d_{ik}d_{ji}d_{jk}^{-1}x^{p_{ii}} \\
\Rightarrow d_{ii}d_{jj}d_{kk}d_{ii} &\not\equiv d_{ij}d_{ji}d_{ik}d_{ki} \Rightarrow \text{(x)}
\end{aligned}
$$

$$
\begin{aligned}
(3.131): d_{ii}x^{p_{ii}} &\not\equiv d_{ik}x^{p_{ik}} \\
\overset{(3.151)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ik}d_{ii}^{-1}d_{ki}d_{kk}^{-1}x^{p_{ii}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.132): d_{ii}x^{p_{ii}} &\not\equiv d_{ij}x^{p_{kj}} \mod (x^n - 1), \\
\overset{(3.148)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-1}d_{jk}d_{jj}^{-1}x^{p_{ik}} \\
\overset{(3.151)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-1}d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{ii}} \\
\Rightarrow d_{ii}d_{jj}d_{kk} &\not\equiv d_{ij}d_{jk}d_{ki} \Rightarrow \text{(ix)}
\end{aligned}
$$

$$
\begin{aligned}
(3.133): d_{ii}x^{p_{ii}} &\not\equiv d_{ij}x^{p_{jj}} \\
\overset{(3.151)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ij}d_{ii}^{-1}d_{ki}d_{kj}^{-1}x^{p_{ii}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.134): d_{ii}x^{p_{ii}} &\not\equiv d_{ik}x^{p_{jk}} \\
\overset{(3.149)}{\Rightarrow} x^{p_{jk}} &\not\equiv d_{ii}d_{ik}^{-1}d_{jk}d_{ji}^{-1}x^{p_{kk}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{jk}} &\not\equiv d_{ii}d_{ik}^{-1}d_{jk}d_{ji}^{-1}d_{ii}d_{ik}^{-1}x^{p_{ki}} \\
\overset{(3.153)}{\Rightarrow} x^{p_{jk}} &\not\equiv d_{ii}d_{ik}^{-1}d_{jk}d_{ji}^{-1} \\
&\quad d_{ii}d_{ik}^{-1}d_{kk}d_{ki}^{-1}x^{p_{jk}} \\
\overset{(ii)}{\Rightarrow} d_{ii}d_{jj}d_{kk} &\not\equiv d_{ik}d_{ji}d_{kj} \Rightarrow \text{(ix)}
\end{aligned}
$$

$$
\begin{aligned}
(3.135): d_{ii}x^{p_{ii}} &\not\equiv d_{ik}x^{p_{kk}} \\
\overset{(3.149)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ik}d_{ii}^{-1}d_{ji}d_{jk}^{-1}x^{p_{ii}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.136): d_{ij}x^{p_{ij}} &\not\equiv d_{ii}x^{p_{ji}} \\
\overset{(3.152)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ii}d_{ij}^{-1}d_{kj}d_{ki}^{-1}x^{p_{ij}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.137): d_{ij}x^{p_{ij}} &\not\equiv d_{ii}x^{p_{ki}} \\
\overset{(3.153)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ii}d_{ij}^{-1}d_{kk}d_{ki}^{-1}x^{p_{jk}} \\
\overset{(3.150)}{\Rightarrow} x^{p_{ii}} &\not\equiv d_{ii}d_{ij}^{-1}d_{kk}d_{ki}^{-1}d_{jj}d_{jk}^{-1}x^{p_{ij}} \\
\Rightarrow d_{ii}d_{jj}d_{kk} &\not\equiv d_{ij}d_{jk}d_{ki} \Rightarrow \text{(ix)}
\end{aligned}
$$

$$
\begin{aligned}
(3.138): d_{ij}x^{p_{ij}} &\not\equiv d_{ik}x^{p_{jk}} \\
\overset{(3.150)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ik}d_{ij}^{-1}d_{jj}d_{jk}^{-1}x^{p_{ij}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,j,k}) \Rightarrow \text{(vi)}
\end{aligned}
$$

$$
\begin{aligned}
(3.139): d_{ij}x^{p_{ij}} &\not\equiv d_{ik}x^{p_{kk}} \\
\overset{(3.150)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ij}d_{ik}^{-1}d_{jk}d_{jj}^{-1}x^{p_{jk}} \\
\overset{(3.153)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ij}d_{ik}^{-1}d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{ki}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ij}d_{ik}^{-1}d_{jk}d_{jj}^{-1} \\
&\quad d_{ki}d_{kk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}} \\
\Rightarrow d_{ii}d_{jj}d_{kk} &\not\equiv d_{ij}d_{jk}d_{ki} \Rightarrow \text{(ix)}
\end{aligned}
$$

Analogously, we consider the cyclically relabelled versions $i \to j \to k \to i$ of the separability conditions (3.126) to (3.139) , with superscript symbol denoting the relabelled versions. Note that (3.137) contradicts (3.150) and is treated separately.

$$
\begin{aligned}
(3.126) : x^{p_{kj}} &\not\equiv x^{p_{ij}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ii}d_{ij}^{-1}x^{p_{ji}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ij}} &\not\equiv d_{ii}d_{ij}^{-1}d_{kj}d_{ki}^{-1}x^{p_{ij}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.127) : x^{p_{jj}} &\not\equiv x^{p_{kj}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{kj}} &\not\equiv d_{ii}d_{ij}^{-1}x^{p_{ki}} \\
\overset{(3.148)}{\Rightarrow} x^{p_{kj}} &\not\equiv d_{ii}d_{ij}^{-1}d_{jj}d_{ji}^{-1}x^{p_{kj}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,j}) \Rightarrow \text{(iv)}
\end{aligned}
$$

$$
\begin{aligned}
(3.128) : x^{p_{jj}} &\not\equiv x^{p_{ij}} \\
\overset{(3.150)}{\Rightarrow} x^{p_{jj}} &\not\equiv d_{jk}d_{jj}^{-1}x^{p_{ki}} \\
\overset{(3.153)}{\Rightarrow} x^{p_{jj}} &\not\equiv d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{ki}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{jj}} &\not\equiv d_{ii}d_{ij}^{-1}d_{ki}d_{kk}^{-1}d_{ij}d_{ii}^{-1}x^{p_{jj}} \\
\Rightarrow d_{ii}d_{jj}d_{kk} &\not\equiv d_{ij}d_{jk}d_{ki} \Rightarrow \text{(ix)}
\end{aligned}
$$

$$
\begin{aligned}
(3.129) : d_{jk}x^{p_{jk}} &\not\equiv d_{ji}x^{p_{ji}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{jk}} &\not\equiv d_{ji}d_{jk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{jk}} \\
&\Rightarrow 0 \not\equiv \mathsf{det}(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}
\end{aligned}
$$

$$
\begin{aligned}
(3.130) : d_{jj}x^{p_{jj}} &\not\equiv d_{jk}x^{p_{jk}} \\
\overset{(3.153)}{\Rightarrow} x^{p_{jj}} &\not\equiv d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{ki}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{jj}} &\not\equiv d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}d_{ij}d_{ii}^{-1}x^{p_{jj}} \\
\Rightarrow d_{ii}d_{jj}d_{kk} &\not\equiv d_{ij}d_{jk}d_{ki} \Rightarrow \text{(ix)}
\end{aligned}
$$

$$
\begin{aligned}
(3.131) : d_{jj}x^{p_{jj}} &\not\equiv d_{ji}x^{p_{ji}} \\
\overset{(3.147)}{\Rightarrow} x^{p_{ji}} &\not\equiv d_{jj}d_{ji}^{-1}d_{ii}d_{ij}^{-1}x^{p_{ki}} \\
\overset{(3.148)}{\Rightarrow} x^{p_{ji}} &\not\equiv d_{jj}d_{ji}^{-1}d_{ii}d_{ij}^{-1}d_{jj}d_{ji}^{-1}x^{p_{ki}} \\
\overset{(3.146)}{\Rightarrow} x^{p_{ji}} &\not\equiv d_{jj}d_{ji}^{-1}d_{ii}d_{ij}^{-1} \\
&\quad d_{jj}d_{ji}^{-1}d_{ii}d_{ij}^{-1}x^{p_{ji}} \\
\overset{(ii)}{\Rightarrow} d_{ii}d_{jj}d_{kk}d_{kk} &\not\equiv d_{ik}d_{ki}d_{jk}d_{kj} \Rightarrow \text{(x)}
\end{aligned}
$$

$$(3.132)\ : d_{jj}x^{p_{jj}} \not\equiv d_{jk}x^{p_{ik}} \overset{(3.147)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{jj}d_{jk}^{-1}d_{ii}d_{ij}^{-1}x^{p_{ki}} \overset{(3.153)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{jj}d_{jk}^{-1}d_{ii}d_{ij}^{-1}d_{kk}d_{ki}^{-1}x^{p_{jk}}$$
$$d_{ii}d_{jj}d_{kk} \not\equiv d_{ij}d_{jk}d_{ki} \Rightarrow \text{(ix)}$$

$$(3.133)\ : d_{jj}x^{p_{jj}} \not\equiv d_{jk}x^{p_{kk}} \overset{(3.147)}{\Rightarrow} x^{p_{jj}} \not\equiv d_{jk}d_{jj}^{-1}d_{ij}d_{ik}^{-1}x^{p_{jj}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,j,j,k}) \Rightarrow \text{(vi)}$$

$$(3.134)\ : d_{jj}x^{p_{jj}} \not\equiv d_{ji}x^{p_{ki}} \overset{(3.147)}{\Rightarrow} x^{p_{jj}} \not\equiv d_{ji}d_{jj}^{-1}d_{ij}d_{ii}^{-1}x^{p_{jj}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,j,i,j}) \Rightarrow \text{(iv)}$$

$$(3.135)\ : d_{jj}x^{p_{jj}} \not\equiv d_{ji}x^{p_{ii}} \overset{(3.151)}{\Rightarrow} x^{p_{jj}} \not\equiv d_{ji}d_{jj}^{-1}d_{kj}d_{ki}^{-1}x^{p_{jj}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,j,j,k}) \Rightarrow \text{(vi)}$$

$$(3.136)\ : d_{jk}x^{p_{jk}} \not\equiv d_{jj}x^{p_{kj}} \overset{(3.146)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{jj}d_{jk}^{-1}d_{ik}d_{ij}^{-1}x^{p_{jk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,j,j,k}) \Rightarrow \text{(vi)}$$

$$(3.137)\ : d_{jk}x^{p_{jk}} \not\equiv d_{jj}x^{p_{ij}}$$

$$(3.138)\ : d_{jk}x^{p_{jk}} \not\equiv d_{ji}x^{p_{ki}} \overset{(3.153)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{ji}d_{jk}^{-1}d_{kk}d_{ki}^{-1}x^{p_{jk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,k,i}) \Rightarrow \text{(vii)}$$

$$(3.139)\ : d_{jk}x^{p_{jk}} \not\equiv d_{ji}x^{p_{ii}} \overset{(3.151)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{ji}d_{jk}^{-1}d_{kj}d_{ki}^{-1}x^{p_{jj}} \overset{(3.147)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{ji}d_{jk}^{-1}d_{kj}d_{ki}^{-1}d_{ii}d_{ij}^{-1}x^{p_{ki}} \overset{(3.153)}{\Rightarrow} x^{p_{jk}} \not\equiv d_{ji}d_{jk}^{-1}d_{kj} d_{ki}^{-1}d_{ii}d_{ij}^{-1}d_{kk}d_{ki}^{-1}x^{p_{jk}}$$
$$\overset{(i)}{\Rightarrow} d_{ji}d_{ik}d_{kj}d_{ki} \not\equiv d_{ii}d_{kk}d_{ji}d_{kj} \Rightarrow d_{ik}d_{ki} \not\equiv d_{ii}d_{kk} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,i,k,i}) \Rightarrow \text{(iii)}$$

Analogously, we consider the cyclically relabelled versions $i \to k \to j \to i$ of the separability conditions (3.126)* to (3.139)*, with superscript symbol * denoting the relabelled versions. Note that (3.137)* contradicts (3.153) and is treated separately.

$$(3.126)^\star : x^{p_{ik}} \not\equiv x^{p_{jk}} \overset{(3.153)}{\Rightarrow} x^{p_{ik}} \not\equiv d_{ki}d_{kk}^{-1}x^{p_{ki}} \overset{(3.148)}{\Rightarrow} x^{p_{ik}} \not\equiv d_{ki}d_{kk}^{-1}d_{jk}d_{ji}^{-1}x^{p_{ik}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,k,i}) \Rightarrow \text{(vii)}$$

$$(3.127)^\star : x^{p_{kk}} \not\equiv x^{p_{ik}} \overset{(3.148)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ji}d_{jk}^{-1}x^{p_{ki}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ji}d_{jk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,i,k}) \Rightarrow \text{(v)}$$

$$(3.128)^\star : x^{p_{kk}} \not\equiv x^{p_{jk}} \overset{(3.153)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}x^{p_{ki}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,k,i,k}) \Rightarrow \text{(iii)}$$

$$(3.129)^\star : d_{ki}x^{p_{ki}} \not\equiv d_{kj}x^{p_{kj}} \overset{(3.148)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{ki}^{-1}d_{ji}d_{jj}^{-1}x^{p_{ki}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,j,j,k}) \Rightarrow \text{(vi)}$$

$$(3.130)^\star : d_{kk}x^{p_{kk}} \not\equiv d_{ki}x^{p_{ki}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,k,i,k}) \Rightarrow \text{(iii)}$$

$$(3.131)^\star : d_{kk}x^{p_{kk}} \not\equiv d_{kj}x^{p_{kj}} \overset{(3.148)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{kk}^{-1}d_{ji}d_{jj}^{-1}x^{p_{ki}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{kk}^{-1}d_{ji}d_{jj}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}} \Rightarrow d_{ii}d_{jj}d_{kk} \not\equiv d_{ji}d_{kj}d_{ik} \Rightarrow \text{(ix)}$$

$$(3.132)^\star : d_{kk}x^{p_{kk}} \not\equiv d_{ki}x^{p_{ji}} \overset{(3.146)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}d_{ij}d_{ii}^{-1}x^{p_{kj}} \overset{(3.148)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}d_{ij}d_{ii}^{-1}d_{ji}d_{jj}^{-1}x^{p_{ki}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}d_{ij}d_{ii}^{-1}d_{ji} d_{jj}^{-1}d_{ik}d_{ii}^{-1}x^{p_{kk}}$$
$$\Rightarrow d_{ii}d_{jj}d_{kk}d_{ii} \not\equiv d_{ki}d_{ij}d_{ji}d_{ik} \Rightarrow \text{(x)}$$

$$(3.133)^\star : d_{kk}x^{p_{kk}} \not\equiv d_{ki}x^{p_{ii}} \overset{(3.149)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{ki}d_{kk}^{-1}d_{jk}d_{ji}^{-1}x^{p_{kk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,k,i}) \Rightarrow \text{(vii)}$$

$$(3.134)^\star : d_{kk}x^{p_{kk}} \not\equiv d_{kj}x^{p_{ij}} \overset{(3.150)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{kk}^{-1}d_{jk}d_{jj}^{-1}x^{p_{jk}} \overset{(3.153)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{kk}^{-1}d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{jk}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{kk}^{-1}d_{jk}d_{jj}^{-1}d_{ki} d_{kk}^{-1}d_{ik}d_{ii}^{-1}x^{p_{jk}}$$
$$\Rightarrow d_{ii}d_{jj}d_{kk}d_{kk} \not\equiv d_{kj}d_{jk}d_{ki}d_{ik} \Rightarrow \text{(x)}$$

$$(3.135)^\star : d_{kk}x^{p_{kk}} \not\equiv d_{kj}x^{p_{jj}} \overset{(3.147)}{\Rightarrow} x^{p_{kk}} \not\equiv d_{kj}d_{kk}^{-1}d_{ik}d_{ij}^{-1}x^{p_{kk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,k,i}) \Rightarrow \text{(vii)}$$

$$(3.136)^\star : d_{ki}x^{p_{ki}} \not\equiv d_{kk}x^{p_{ik}} \overset{(3.153)}{\Rightarrow} x^{p_{ki}} \not\equiv d_{kk}d_{ki}^{-1}d_{ji}d_{jk}^{-1}x^{p_{ki}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,k,i}) \Rightarrow \text{(vii)}$$

$$(3.137)^\star : d_{ki}x^{p_{ki}} \not\equiv d_{kk}x^{p_{jk}}$$

$$(3.138)^\star : d_{ki}x^{p_{ki}} \not\equiv d_{kj}x^{p_{ij}} \overset{(3.150)}{\Rightarrow} x^{p_{ki}} \not\equiv d_{kj}d_{ki}^{-1}d_{jk}d_{jj}^{-1}x^{p_{jk}} \overset{(3.153)}{\Rightarrow} x^{p_{ki}} \not\equiv d_{kj}d_{ki}^{-1}d_{jk}d_{jj}^{-1}d_{ki}d_{kk}^{-1}x^{p_{jk}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{k,j,k,j}) \Rightarrow \text{(viii)}$$

$$(3.139)^\star : d_{ki}x^{p_{ki}} \not\equiv d_{kj}x^{p_{jj}} \overset{(3.147)}{\Rightarrow} x^{p_{ki}} \not\equiv d_{kj}d_{ki}^{-1}d_{ii}d_{ij}^{-1}x^{p_{ki}} \Rightarrow 0 \not\equiv \det(\boldsymbol{D}_{i,j,i,k}) \Rightarrow \text{(v)}$$

The constraints (i) and (ii) are proven as follows.

$$(3.147) : d_{ij}x^{p_{jj}} \equiv d_{ik}x^{p_{kk}} \overset{(3.148)}{\Rightarrow} d_{ij}x^{p_{jj}} \equiv d_{ik}d_{ji}d_{jk}^{-1}x^{p_{ii}} \overset{(3.151)}{\Rightarrow} d_{ij}x^{p_{jj}} \equiv d_{ik}d_{ji}d_{jk}^{-1}d_{kj}d_{ki}^{-1}x^{p_{jj}} \Rightarrow d_{ij}d_{jk}d_{ki} \equiv d_{ik}d_{kj}d_{ji} \Rightarrow \text{(i)}$$

$$(3.146) : d_{ik}x^{p_{jk}} \equiv d_{ii}x^{p_{ji}} \overset{(3.150)}{\Rightarrow} d_{ik}d_{jj}d_{jk}^{-1}x^{p_{ij}} \equiv d_{ii}x^{p_{ji}} \overset{(3.149)}{\Rightarrow} d_{ik}d_{jj}d_{jk}^{-1}x^{p_{ij}} \equiv d_{ii}d_{kj}d_{ki}^{-1}x^{p_{ij}} \Rightarrow d_{jj}d_{ik}d_{ki} \equiv d_{ii}d_{jk}d_{kj} \Rightarrow \text{(ii)}$$

$$(3.147) : d_{ii}x^{p_{ki}} \equiv d_{ij}x^{p_{jj}} \overset{(3.151)}{\Rightarrow} d_{ii}x^{p_{ki}} \equiv d_{ij}d_{kk}d_{kj}^{-1}x^{p_{ik}} \overset{(3.148)}{\Rightarrow} d_{ii}x^{p_{ki}} \equiv d_{ij}d_{kk}d_{kj}^{-1}d_{ji}d_{jk}^{-1}x^{p_{ki}} \Rightarrow d_{ii}d_{jk}d_{kj} \equiv d_{kk}d_{ij}d_{ji} \Rightarrow \text{(ii)}$$

$$(3.148) : d_{jj}x^{p_{kj}} \equiv d_{ji}x^{p_{ki}} \overset{(3.153)}{\Rightarrow} d_{jj}x^{p_{kj}} \equiv d_{ji}d_{kk}d_{ki}^{-1}x^{p_{jk}} \overset{(3.146)}{\Rightarrow} d_{jj}x^{p_{kj}} \equiv d_{ji}d_{kk}d_{kj}^{-1}d_{ij}d_{ik}^{-1}x^{p_{kj}} \Rightarrow d_{ii}d_{jk}d_{kj} \equiv d_{kk}d_{ij}d_{ji} \Rightarrow \text{(ii)}$$

Since all separability conditions hold and yield the conditions as given in Theorem 3.12, the proof is complete.

B Example for Cyclic Interference Neutralization on the $2 \times 2 \times 2$ Relay-Interference Channel

Let us consider a simple example for cyclic IN on the $2\times2\times2$ relay interference channel with $n = 4$ dimensions. We choose the following two valid channel matrices for the UL and DL subchannels:

$$\boldsymbol{D} = \begin{pmatrix} x^3 & x^3 \\ x^1 & x^0 \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} x^2 & x^0 \\ x^2 & x^3 \end{pmatrix}.$$

By checking the conditions in Corollary 4.3, we see that they are fulfilled:

$$\begin{aligned}
\delta_{12} + \delta_{21} + \eta_{11} + \eta_{22} \equiv \delta_{11} + \delta_{22} + \eta_{12} + \eta_{21} &\equiv 1 \pmod 4, \\
\delta_{12} + \delta_{21} + \eta_{12} + \eta_{21} \equiv 2 \not\equiv \delta_{11} + \delta_{22} + \eta_{11} + \eta_{22} &\equiv 0 \pmod 4, \\
\det(\boldsymbol{D}) \equiv x^3 - x^4 \not\equiv 0 \bmod (x^4 - 1)&, \\
\det(\boldsymbol{E}) \equiv x^1 - x^2 \not\equiv 0 \bmod (x^4 - 1)&.
\end{aligned}$$

Next, we consider the interference-neutralization conditions in (4.12) and (4.13) for the parameters $\gamma_1 = 0$ and $\gamma_2 = 1$ at R_1 and R_2:

$$\begin{aligned}
\delta_{12} + \gamma_1 + \eta_{11} \equiv \delta_{22} + \gamma_2 + \eta_{12} \equiv 1 \pmod 4, \\
\delta_{11} + \gamma_1 + \eta_{21} \equiv \delta_{21} + \gamma_2 + \eta_{22} \equiv 1 \pmod 4.
\end{aligned}$$

Furthermore, the no-signal-neutralization conditions (4.14) and (4.15) hold as well:

$$\delta_{11} + \gamma_1 + \eta_{11} \equiv 1 \not\equiv \delta_{21} + \gamma_2 + \eta_{12} \equiv 2 \pmod 4,$$
$$\delta_{12} + \gamma_1 + \eta_{21} \equiv 1 \not\equiv \delta_{22} + \gamma_2 + \eta_{22} \equiv 0 \pmod 4.$$

By choosing parameter $\tau = 3$, we obtain the following transmission signals for Tx_1 and Tx_2:

$$u_1(x) = W_1^{[0]}x^0 + W_1^{[1]}x^1 + W_1^{[2]}x^2 + W_1^{[3]}x^3,$$
$$u_3(x) = W_2^{[0]}x^0 + W_2^{[1]}x^1 + W_2^{[2]}x^2.$$

The received signals at the relays R_1 and R_2 are:

$$r_1(x) \equiv (W_1^{[1]} + W_2^{[1]})x^0 + (W_1^{[2]} + W_2^{[2]})x^1 + W_1^{[3]}x^2 + (W_1^{[0]} + W_2^{[0]})x^3 \bmod (x^4 - 1),$$
$$r_2(x) \equiv (W_1^{[3]} + W_2^{[0]})x^0 + (W_1^{[0]} + W_2^{[1]})x^1 + (W_1^{[1]} + W_2^{[2]})x^2 + W_1^{[2]}x^3 \bmod (x^4 - 1).$$

With parameter $k_2 = \tau + \delta_{22} \equiv 3 \pmod 4$, we set $r_2^{[3]} = 0$. The resulting forwarded signals, with the given parameters $\gamma_1 = 0$ and $\gamma_2 = 1$, are:

$$v_1(x) \equiv (W_1^{[1]} + W_2^{[1]})x^0 + (W_1^{[2]} + W_2^{[2]})x^1 + W_1^{[3]}x^2 + (W_1^{[0]} + W_2^{[0]})x^3 \bmod (x^4 - 1),$$
$$v_2(x) \equiv (-W_1^{[3]} - W_2^{[0]})x^1 + (-W_1^{[0]} - W_2^{[1]})x^2 + (-W_1^{[1]} - W_2^{[2]})x^3 \bmod (x^4 - 1),$$

so that the following signals are received at the destinations Rx_1 and Rx_2:

$$\begin{aligned}
t_1(x) \equiv\; & W_1^{[3]}x^0 + (W_1^{[0]} + W_2^{[0]} - W_1^{[3]} - W_2^{[0]})x^1 + \\
& (W_1^{[1]} + W_2^{[1]} - W_1^{[0]} - W_2^{[1]})x^2 + (W_1^{[2]} + W_2^{[2]} - W_1^{[1]} - W_2^{[2]})x^3 \\
\equiv\; & W_1^{[3]}x^0 + (W_1^{[0]} - W_1^{[3]})x^1 + (W_1^{[1]} - W_1^{[0]})x^2 + (W_1^{[2]} - W_1^{[1]})x^3 \bmod (x^4 - 1), \\
t_2(x) \equiv\; & (W_1^{[3]} - W_1^{[3]} - W_2^{[0]})x^0 + (W_1^{[0]} + W_2^{[0]} - W_1^{[0]} - W_2^{[1]})x^1 + \\
& (W_1^{[1]} + W_2^{[1]} - W_1^{[1]} - W_2^{[2]})x^2 + (W_1^{[2]} + W_2^{[2]})x^3 \\
\equiv\; & -W_2^{[0]}x^0 + (W_2^{[0]} - W_2^{[1]})x^1 + (W_2^{[1]} - W_2^{[2]})x^2 + (W_1^{[2]} + W_2^{[2]})x^3 \bmod (x^4 - 1).
\end{aligned}$$

At both destinations, all interfering signals are neutralized. It is obvious that the submessages $W_1^{[0]}$, $W_1^{[1]}$, $W_1^{[2]}$, $W_1^{[3]}$ are linear decodable at Rx_1, and that the submessages $W_2^{[0]}$, $W_2^{[1]}$, $W_2^{[2]}$ are linear decodable at Rx_2. Hence, $\frac{2n-1}{n} = \frac{7}{4}$ DoF are achieved for $n = 3$ in accordance to Theorem 4.2.

C Genie-Aided Upper Bounds of the $\mathrm{D3C}(n_1, n_2, n_3)$ with $n_3 \geq n_2 \geq n_1$

The remaining upper bounds on the capacity region of the $\mathrm{D3C}(n_1, n_2, n_3)$ are derived similar to Section 5.2.2. We only discuss the main differences in this appendix.

(i): To derive the bound $R_{12} + R_{13} + R_{23} \leq n_3$, we provide W_{32} as side-information to the receiver of T_1, and proceed similar to Section 5.2.2. That is, we prove that the enhanced T_1 can construct $\boldsymbol{y}_2^N$ from which it can decode W_{23}.

(ii): To derive the bound $R_{21} + R_{23} + R_{13} \le n_3 + n_2 - n_1$, we provide W_{31} and $\hat{\boldsymbol{x}}_3^N$ to T_2 as side-information, where $\hat{\boldsymbol{x}}_3^N$ denotes the lowermost $n_2 - n_1$ bits of $\boldsymbol{x}_3^N$. Providing these bits is necessary since T_2 can only obtain the topmost n_1 symbols of $\boldsymbol{x}_3(1)$ from $\boldsymbol{y}_2(1)$ and $\boldsymbol{x}_1(1)$ after decoding W_{21} (recall that given W_{31} and W_{21}, T_2 can construct $\boldsymbol{x}_1(1)$). By combining the topmost n_1 bits of $\boldsymbol{x}_3(1)$ and the lowermost $n_2 - n_1$ bits provided by the side-information, T_2 can construct $\boldsymbol{y}_1(1)$ and hence also $\boldsymbol{x}_1(2)$, since it knows W_{31} from side-information and W_{21} after decoding. Similarly, all components of $\boldsymbol{y}_1^N$ can be constructed, and W_{13} can be decode. Thus, by using Fano's inequality, we can write:

$$
\begin{aligned}
& N(R_{21} + R_{23} + R_{13}) \\
& \le I(W_{21}, W_{23}, W_{13}; \boldsymbol{y}_2^N, \hat{\boldsymbol{x}}_3^N, W_{12}, W_{32}, W_{31}) + N\epsilon_N \\
& \le \mathrm{H}(\boldsymbol{y}_2^N, \hat{\boldsymbol{x}}_3^N) - \mathrm{H}(\boldsymbol{y}_2^N, \hat{\boldsymbol{x}}_3^N | \boldsymbol{w}) + N\epsilon_N \\
& \le \mathrm{H}(\boldsymbol{y}_2^N, \hat{\boldsymbol{x}}_3^N) + N\epsilon_N \\
& \le N(\max(n_3 + n_2 - n_1, n_1 + n_2 - n_1) + \epsilon_N) \\
& = N(n_3 + n_2 - n_1 + \epsilon_N),
\end{aligned}
$$

where $\epsilon_N \to 0$ as $N \to \infty$. This provides $R_{21} + R_{23} + R_{13} \le n_3 + n_2 - n_1$ after dividing by N and letting $N \to \infty$.

(iii): To derive $R_{21} + R_{23} + R_{31} \le n_3$, we provide W_{13} to T_2 and proceed similar to **(i)**, by showing that T_2 can construct $\boldsymbol{y}_3^N$ and decode W_{31}.

(iv): To derive $R_{31} + R_{32} + R_{21} \le n_3$, we give $\hat{\boldsymbol{x}}_1^N$ and W_{12} to T_3 as side-information, where $\hat{\boldsymbol{x}}_1^N$ denotes the lowermost $n_3 - n_2$ bits of $\boldsymbol{x}_1^N$. By proceeding similar to **(ii)**, we can show that T_3 can construct $\boldsymbol{y}_2^N$ given this side-information, and then decode W_{21}, leading to the desired bound.

(v): To derive the bound $R_{31} + R_{32} + R_{12} \le n_3 + n_2 - n_1$, we give W_{21} and $\hat{\boldsymbol{x}}_2^N$ to T_3, where $\hat{\boldsymbol{x}}_2^N$ denote the lowermost $n_3 - n_1$ bits of $\boldsymbol{x}_2^N$. Similar to **(ii)**, T_3 is able to construct $\boldsymbol{y}_1^N$ given this side-information, and then to decode W_{12}, leading to the desired bound.

As a result, we obtain the upper bounds on capacity region as given by (5.47) to (5.54).

List of Acronyms

2IFC	2-user interference channel
AWGN	additive white Gaussian noise
BC	broadcast channel
BD	backward decoding
BHN	backhaul network
CPCM	cyclic polynomial channel model
CSI	channel state information
D2D	device-to-device
DL	downlink
DoF	degrees-of-freedom
FF	feedforward
GDoF	generalized degrees-of-freedom
GMCM	Gaussian MIMO channel model
GPS	global positioning system
IA	interference alignment
IAC	interference alignment and cancellation
IFWD	interference forwarding
IN	interference neutralization
LDCM	linear deterministic channel model
LEaD	linear encoding and decoding
LoS	line-of-sight
MA	multiple-access
MAC	multiple-access channel
MIMO	multiple-input multiple-output
OFDM	orthogonal frequency division multiplexing
SA	signal alignment
SISO	single-input single-output
SNR	signal-to-noise-ratio
UL	uplink

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