Compressed Sensing - beyond the Shannon Paradigm

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joint work with Albert Cohen and Ron DeVore

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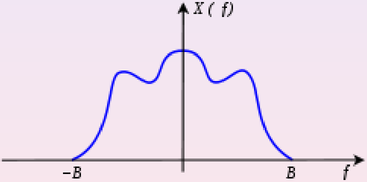

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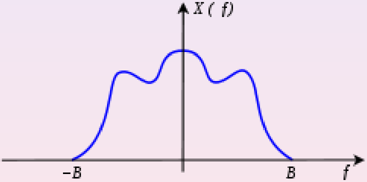

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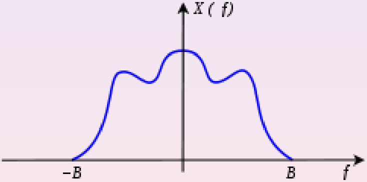

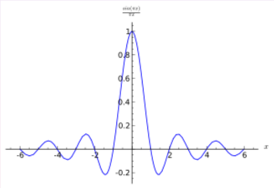

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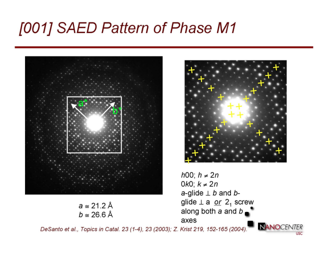
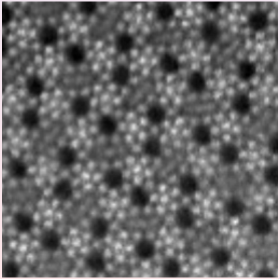
- ...they are far from bandlimited
- ...however, they are sparse, i.e. their information content is small compared with their size
- ...measurements are difficult/expensive/harmful

(Information-)Sparse Signals

...are those sparse?





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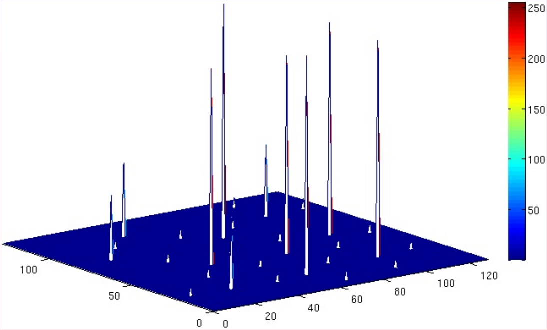
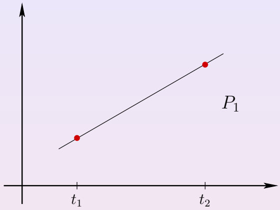


An Instructive Example: Polynomials



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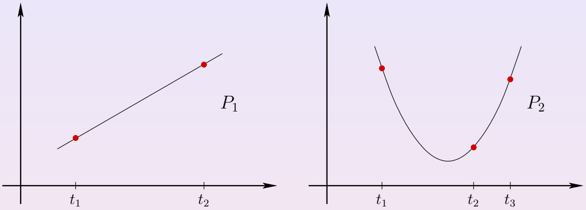

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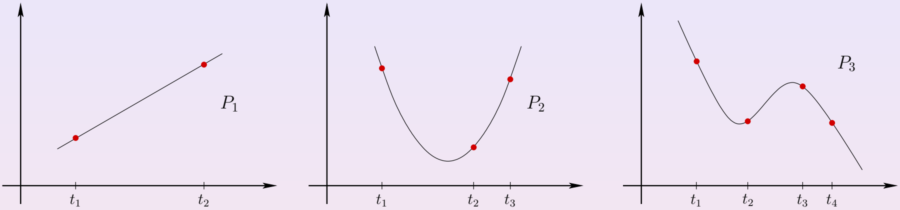

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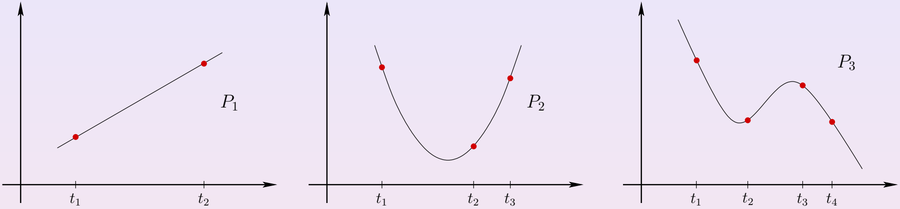

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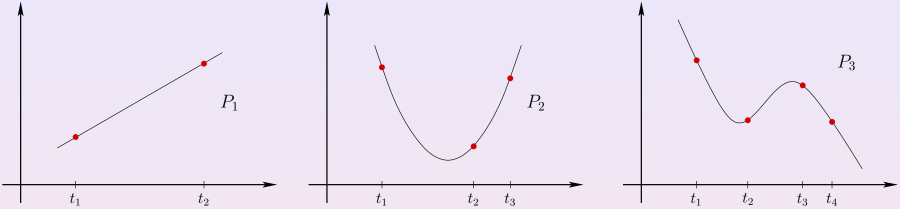

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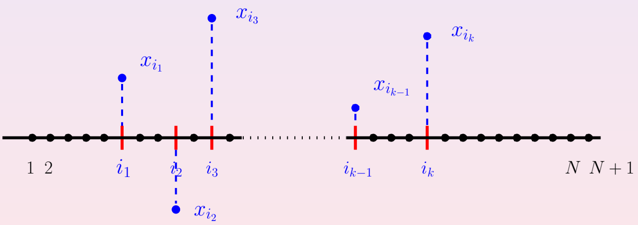

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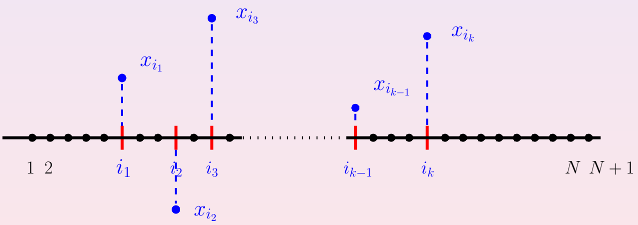

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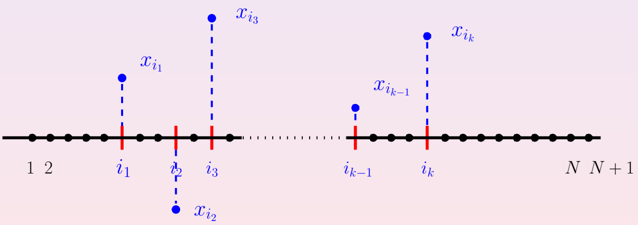

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The Mathematical Model

- $x = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$ "large" signal

Candés/Romberg/Tao, Donoho, Gilbert/Strauss/Tropp, Tanner, Rice group, Needell/Vershynin, CDD...

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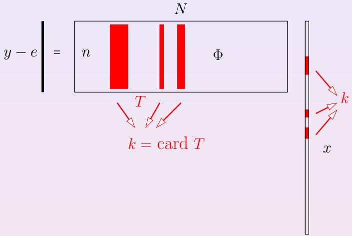

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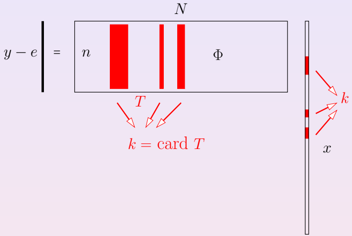

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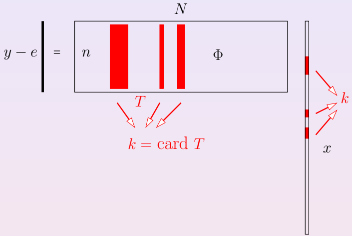

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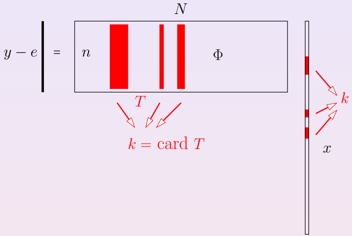

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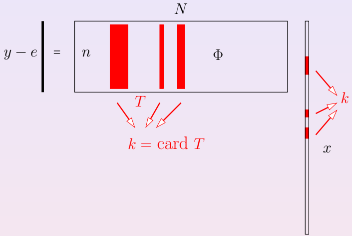

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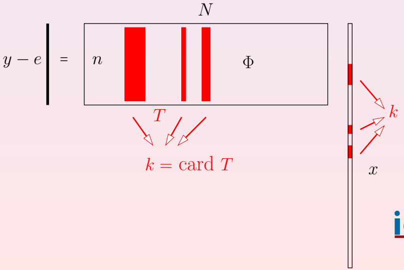

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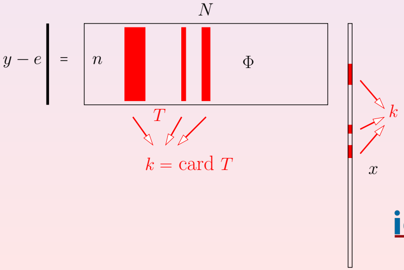

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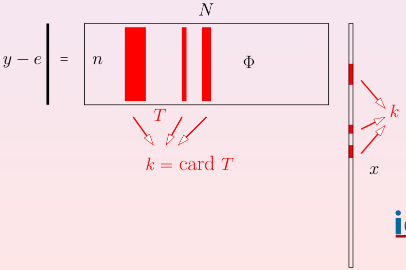

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Proof of: (i)⇒(ii): Suppose $x \in \Sigma_{2k} \cap \ker \Phi$

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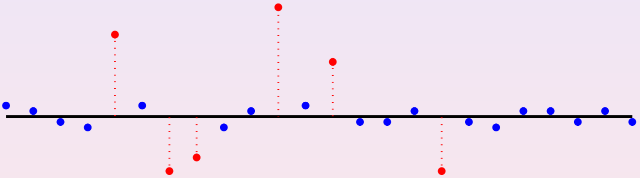

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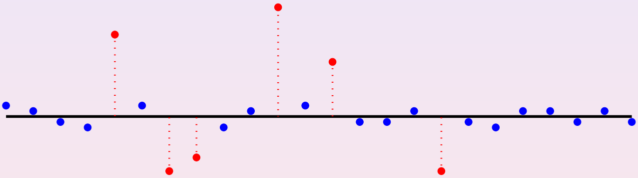

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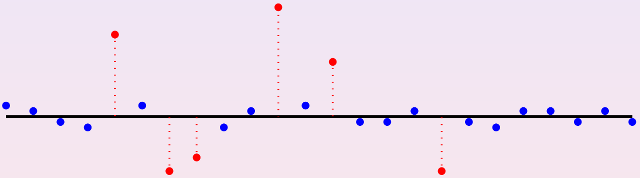

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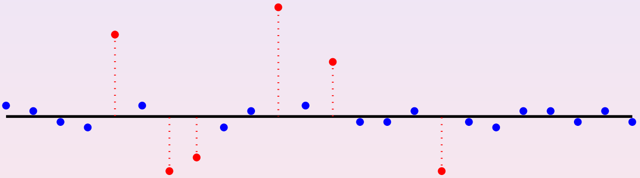

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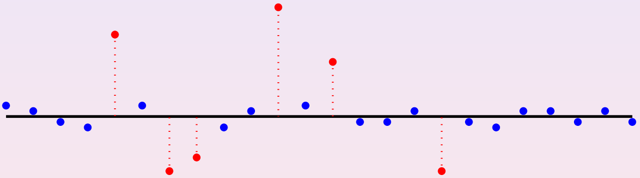

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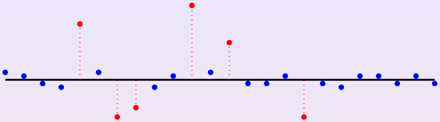

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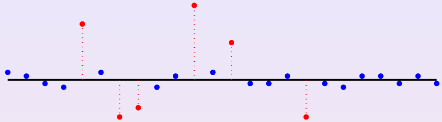

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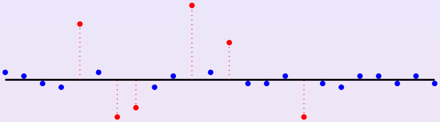

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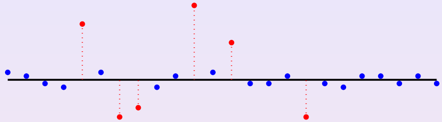

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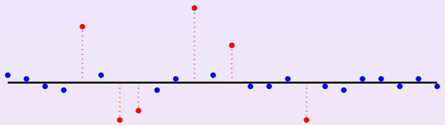

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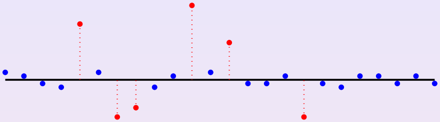

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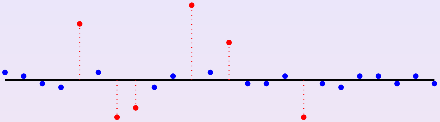

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Maximal Optimality Range k? - back to the 70's

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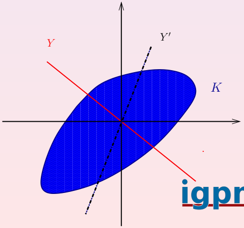

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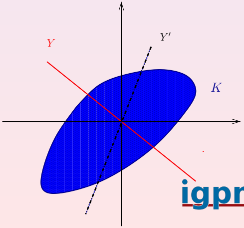

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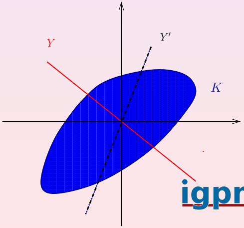

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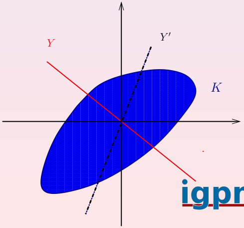

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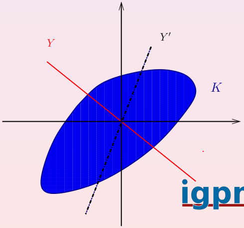

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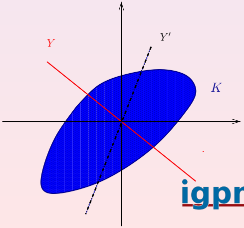

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THEOREM [CT]: Praktikable decoder– ℓ_1-minimization

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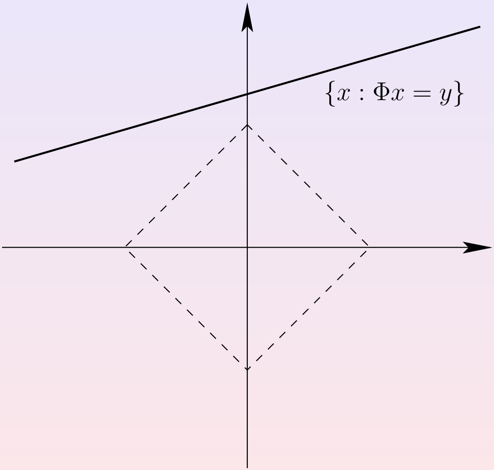
$$\Longrightarrow \|x - \Delta(\Phi x)\|_{\ell_2} \lesssim \frac{\sigma_k(x)_{\ell_1}}{\sqrt{k}} \quad k\text{-sparse exact}$$

Convex relaxation: $\min \|\cdot\|_{\ell_0} \rightarrow \min \|\cdot\|_{\ell_1}$

A Geometric Explanation - $n = 1, N = 2$



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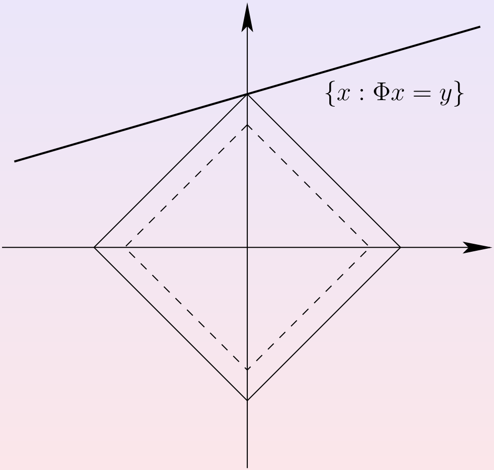
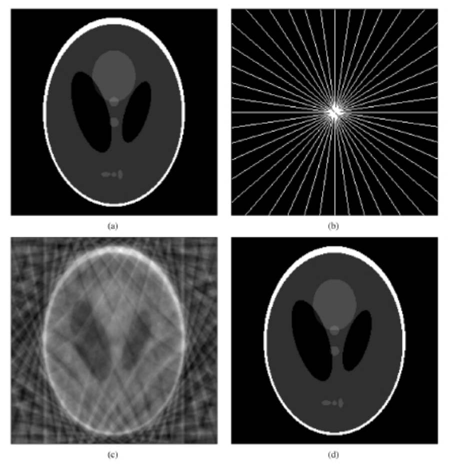


Closely Related: TV-Minimization... - Δ?



(Candes/Romberg/Tao)

How to Realize RIP? - Randomness Helps.. - Φ?

The “Rice - One Pixel - Camera”

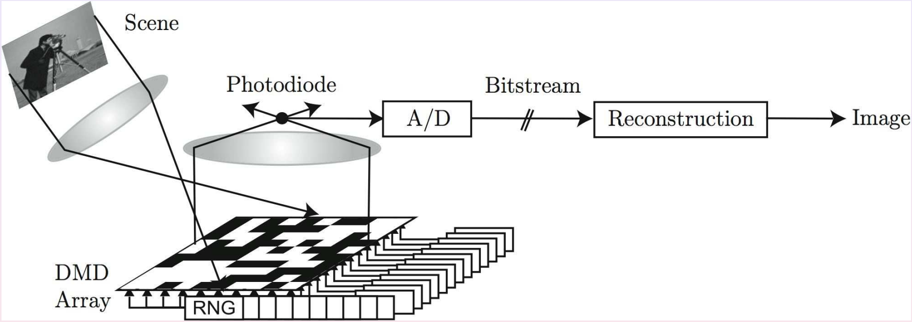


Figure: (R. Baraniuk, Rice University)

Randomness Helps...Concentration of Measure Property

Gaussian matrices: $\phi_{i,j} \in \mathcal{N}(0, 1/n)$

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Gauß-, Bernoulli-, uniformly distributed points on the $(n-1)$-sphere satisfy **CoMP**

The Norm Matters...

THEOREM: $\|x\|_{\ell_1} = \sum_{j=1}^{N} |x_j|$:

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- Assume that $\{\Phi(\omega)\}$ satisfies CoMP. Then $\exists\ \Delta$ s.t. for each $x \in \mathbb{R}^N$ with high probability (with respect to .Φ) one has

$$\|x - \Delta(\Phi x)\|_{\ell_2} \le C_0\sigma_k(x)_{\ell_2}, \quad k \lesssim n/\log(N/n)$$

An Optimal Decoder

Decoder:

(i) Set $r^0 := y$, $x^0 := 0$, $j = 0$, $\Lambda_0 = \bar{\Lambda}_0 = \emptyset$

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take $x^{j+1} =$ best k-term approximation of $\hat{x}$ und $\Lambda_{j+1} := \mathrm{supp}\, x^{j+1}$, set $j + 1 \to j$, and go to (ii).

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Let $\delta \leq 1/16$ and let $\{\Phi = \Phi(\omega)\}$ satisfy **CoMP**. If $y = \Phi x + e$, then

$$\|x - x^*\|_{\ell_2} \leq C(\sigma_k(x)_{\ell_2} + \|e\|_{\ell_2}), \quad k \leq an/\log(N/n)$$

for all draws from $\Omega^* \subset \Omega$ of measure at least $1 - N^{-\beta}$, where $a \searrow$ when $\beta \nearrow$.

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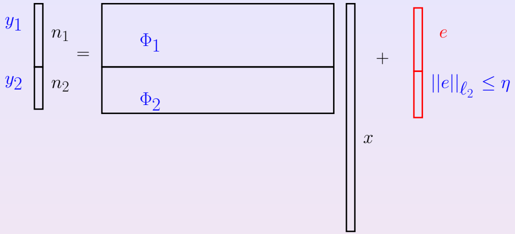
- Thresholding is more efficient, but requires smaller δ
- ℓ_1-minimimization/regularization is more robust but more expensive...

igpm

ℓ_1-Regularization



$$\begin{aligned} y_1 &= \Phi_1 x + e_1 \\ y_2 &= \Phi_2 x + e_2 \end{aligned}$$

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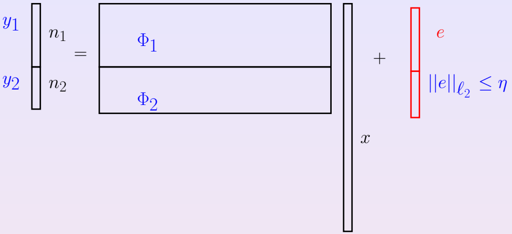

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1. For $j = 1, \cdots, J$ set $\epsilon_j = 2^{2-j}\|y\|_{\ell_2}$, and solve

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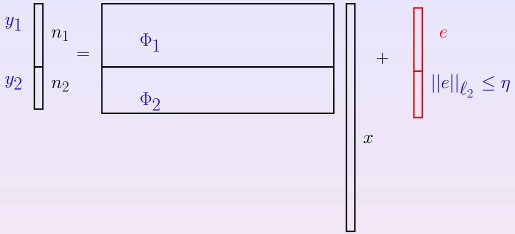

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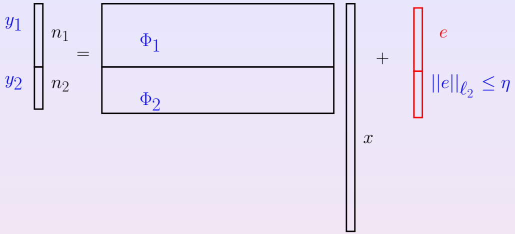

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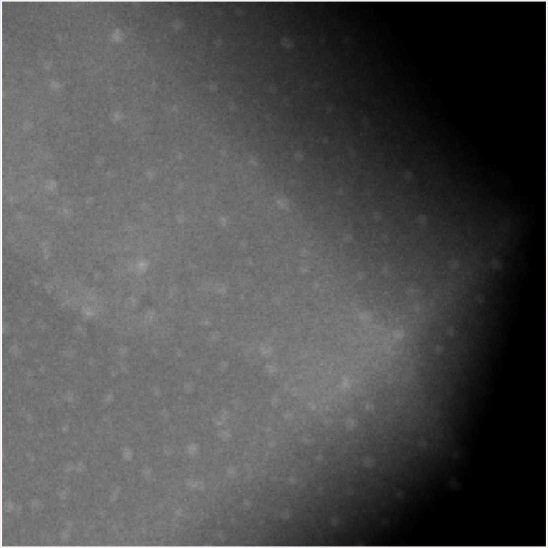
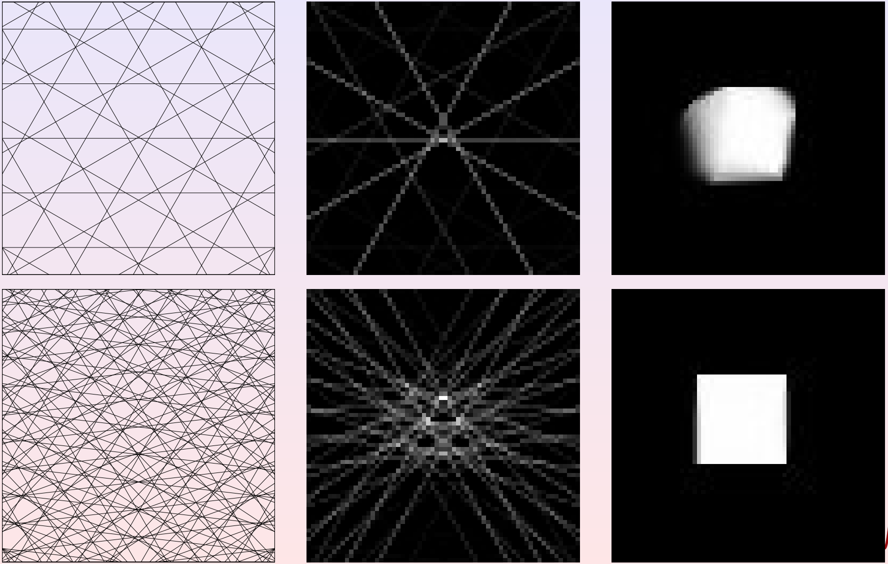
with probability larger than $1 - t(n_1, n_2)$ where

$$t(n_1, n_2) = Jb_2 e^{-c_2 n_2/2} + b_1(e^{-n_1 \frac{c_1}{32}} + e^{-c_1 n_1} + e^{-c_1 n_1/2})$$

Electron-Tomography



Tests



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