Rateless Codes Based on Punctured Polar Codes

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***Abstract*—Polar codes are the first binary linear block codes provably achieving the symmetric capacity of arbitrary binary-input discrete memoryless channels. However, in their original design, their block length is limited to integer powers of two, a constraint that may be relaxed by puncturing. In this work, a novel construction of rateless codes based on punctured polar codes is presented, that is, codes that offer flexible rates via length adaption for a fixed dimension. While the approach presented relies on puncturing, it may be based on arbitrary puncturing methods. The rateless codes obtained work with standard polar code encoders and decoders, and allow for ad-hoc switching of the code rate without additional overhead.**

I. Introduction

Polar codes (PCs) provide the first channel coding scheme provably achieving the symmetric capacity of arbitrary binary-input discrete memoryless channels (BDMCs). Thus, they have received a lot of attention since their presentation [1]. By providing explicit constructions, as well as low-complexity encoding and decoding methods under which the symmetric capacity of any BDMC is attained asymptotically, PCs give an answer to a long-standing open question in information theory. However, in their original form based on the two-dimensional polarization kernel

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{1}$$

as in [1], possible block lengths are restricted to integer powers of two.

Adaptions of the original PC construction have been proposed to facilitate greater length flexibility as required in many modern communication applications. As a first option, the two-dimensional kernel $\mathbf{F}$ as in (1) may be replaced by other kernels $\mathbf{K} \in \mathbb{F}_2^{m \times m}$ [2], resulting in PCs of lengths m^n for some $n \in \mathbb{N}$, where $\mathbb{F}_2 := \{0, 1\}$ denotes the binary finite field. Another approach is given by concatenated coding schemes including PCs as inner or outer codes. Various such schemes have been presented in the literature, concatenating PCs with, *e.g.*, Reed-Solomon (RS) codes [3], [4], low-density parity-check (LDPC) codes [5], or Bose-Chaudhuri-Hocquenghem (BCH) codes [6].

However, these modifications require substantial changes of both the encoder and decoder structures. In addition to that, neither of them is able to enhance length flexibility of PC schemes to the desired extent. As a result, both puncturing and shortening of PCs based on the standard kernel $\mathbf{F}$ have been considered, allowing for fine-grained length adaptions of PCs, *cf.*, *e.g.*, [5], [7], or [8].

In this work, we present a novel approach of constructing rateless codes based on puncturing PCs which may be employed with standard successive cancellation (SC) decoding as in [1] or successive cancellation list (SCL) decoding [9]. Such rateless code will be given by a sequence of linear codes of identical dimension but increasing lengths and thus decreasing rates, such that each instance may be obtained by appropriately puncturing a longer code of the sequence. As a result, this requirement imposes constraints on the sequence of puncturing patterns used for construction. Additional constraints result from requiring compatibility with standard decoders, facilitating low-complexity decoding of the proposed constructions. In addition to that, the construction approach is generic in the sense that it may be based on arbitrary puncturing approaches.

Rateless codes provide a crucial building block for hybrid automatic repeat request with incremental redundancy (HARQ-IR) schemes. HARQ-IR schemes present a way to obtain capacity-achieving throughput efficiency, albeit potentially introducing additional delays due to necessary retransmissions [10], [11], [12]. Such schemes prove helpful in scenarios where channel quality is fluctuating, or when only imperfect, *e.g.*, outdated, channel-state information (CSI) is available.

In general, codes for use in HARQ-IR schemes facilitate adapting the number of parity bits sent, hence adapting the code length N depending on the quality of the underlying channel, for a fixed amount of K information bits. As a result, via adjusting N, the rate of the channel code used may be adapted to varying channel conditions. Furthermore, in case of retransmissions due to decoding failures at the receiver, only additional coded bits have to be provided, incrementally equipping the channel decoder with additional redundancy to enable successful decoding.

A first HARQ-IR scheme based on PCs is presented in [10], building on the quasi-uniform puncturing (QUP) heuristic presented in [13]. Another HARQ-IR scheme presented in [14] builds on a rate-compatible family of PCs obtained by both puncturing and extending a mother PC. Despite their simplicity, both schemes show good throughput performance in the simulations reported.

Other HARQ-IR schemes relying on PCs focusing on achieving capacity on certain families of BDMCs have been proposed. This includes the work of Li *et al*. [15], which achieves the capacity of a general class of BDMCs totally ordered by stochastic degradation, but adapts rates by adjusting the dimension of the codes, and not by adapting the length via, *e.g.*, puncturing. In a similar fashion, [16] and [17] present schemes that allow for incremental retransmissions of variable lengths.

This paper is structured as follows. In Section II, we discuss preliminaries of PCs and the puncturing approach used in this work. Section III presents the proposed construction of rateless codes based on puncturing PCs. We provide an example of a rateless code constructed by the approach, and evaluate its performance in Section IV. Section V concludes the paper.

II. Preliminaries

A. Polar Codes

In [1], Arıkan presents a method to transform $N = 2^n$, $n \in \mathbb{N}$, independent copies of a BDMC $\mathcal{W} : \mathcal{X} \to \mathcal{Y}$, with input alphabet $\mathcal{X} = \{0,1\}$, and output alphabet $\mathcal{Y}$, into a set of N (virtual) channels $\{\mathcal{W}_{N,i} : i \in [N]\}$, where we write $[N] := \{1,\ldots,N\} \subset \mathbb{N}$. This results in channels

$$\mathcal{W}_{N,i} : \mathcal{X} \to \mathcal{Y}^N \times \mathcal{X}^{i-1}, \tag{2}$$

also referred to as coordinate channels. These coordinate channels $\mathcal{W}_{N,i}$ polarize in the sense that as N approaches infinity, the fraction of indices i vanishes for which $I(\mathcal{W}_{N,i})$ neither approaches 0 (useless channels) nor 1 (perfect channels). Here, we follow [1] and write $I(\mathcal{W}_{N,i})$ to denote the symmetric capacity of the channel $\mathcal{W}_{N,i}$, *i.e.*, its mutual information assuming uniformly distributed channel inputs. This effect is referred to as channel polarization (CP) [1].

As N approaches infinity through powers of two, the fraction of indices i such that $I(\mathcal{W}_{N,i}) \to 1$ is arbitrarily close to $I(\mathcal{W})$ [1]. If $\mathcal{W}$ is an output-symmetric channel, the symmetric capacity $I(\mathcal{W})$ is equal to its Shannon capacity. Hence, PCs provide a coding scheme which is capacity-achieving on the family of output-symmetric BDMCs.

To polarize N independent copies of a basic channel $\mathcal{W}$ used for transmission, these copies are combined recursively by linear operations on the information to transmit. Hence, this recursion results in a linear code $\mathcal{C}$, which forms codewords containing N linear combinations

$$\mathbf{x} = (x_1, \ldots, x_N) = \mathbf{u}_{\mathcal{A}} \mathbf{G}_{N,\mathcal{A}} \tag{3}$$

of $\mathbf{u} \in \mathbb{F}_2^N$, where $\mathbf{u}_{\mathcal{A}} = (u_i : i \in \mathcal{A})$ denotes a subvector of $\mathbf{u}$ containing the information bits, and $\mathcal{A} \subset [N]$ is the index set providing the indices of these information positions.

A generator matrix $\mathbf{G}_{N,\mathcal{A}}$ of the code is obtained by selecting rows $\mathbf{g}_i$, $i \in \mathcal{A}$ from $\mathbf{G}_N = \mathbf{B}_N \mathbf{F}^{\otimes n}$. Here, $\mathbf{B}_N$ is a bit-reversal permutation, and $\mathbf{F}$ as in (1) is the default binary polarization kernel [1]. This assigns fixed zero values to the remaining elements $\mathbf{u}_{\mathcal{A}^c}$, where $\mathcal{A}^c = [N] \setminus \mathcal{A}$ denotes the complement of $\mathcal{A}$ with respect to $[N]$. These fixed positions are referred to as *frozen bits*. Choosing $\mathcal{A}$ such that $|\mathcal{A}| = K$, we obtain a code of rate $R = \frac{K}{N}$.

B. Puncturing Polar Codes

Assuming a mother PC $\mathcal{C}$ of length N, a punctured PC $\mathcal{C}'$ of length $M < N$ is defined by a puncturing pattern $\mathcal{P} \subset [N]$ such that $P := |\mathcal{P}| = N - M$. Finding an optimal puncturing pattern $\mathcal{P}$ of cardinality $P = N - M$ for a given mother PC of length N and dimension K for targeting a desired length $M < N$ is an open problem [8], [18]. Consequently, various heuristics have been proposed.

Puncturing of PCs was first considered in [5]. To optimize performance of the punctured code under belief-propagation (BP) decoding, the authors suggest to puncture positions connected to the fewest number of stopping trees. A second heuristic for constructing $\mathcal{P}$ is given by choosing $N - M$ positions from $[N]$ uniformly at random, as used in [8], [7], and [19].

Another method referred to as QUP is presented in [13]. To obtain an incidence vector describing the puncturing pattern by this approach, the authors suggest to shuffle a vector containing an appropriate number of ones via the bit-reversal permutation $\mathbf{B}_N$. More formally, to construct a QUP pattern $\mathcal{P}$ of cardinality P, an incidence vector $\mathbf{p}$ of $\mathcal{P}$ is constructed as [13]

$$\mathbf{p} = (\underbrace{1, \ldots, 1}_{\text{first } P \text{ positions}}, 0, \ldots, 0) \cdot \mathbf{B}_N. \tag{4}$$

Hence, we have

$$\mathcal{P} = \{j : p_j = 1\} \subset [N]\,. \tag{5}$$

For the constructions presented in this work we rely on QUP, as it provides patterns resulting in very good code performance, a claim supported by simulation results presented in [18].

C. Construction

To construct a PC of a desired length N, an index set $\mathcal{A} \subseteq [N]$ indicating the positions used for information bits has to be selected, in order to minimize the union bound on block error under SC decoding [1].

An explicit recursion in the Bhattacharyya parameters of the coordinate channels is given by Arıkan in [1], assuming the basic channel $\mathcal{W}$ is a binary erasure channel (BEC). However, as the output alphabets of the channels $\mathcal{W}_{N,i}$ as given in (2) grow exponentially in N, exact construction is of exponential complexity for general channels. As a result, other approaches are necessary to construct PCs for arbitrary BDMCs.

In [20], Mori and Tanaka suggest to estimate the bit decision error probabilities $\mathbb{P}[\hat{u}_i \neq u_i]$ under SC decoding by tracking corresponding message densities by a density evolution (DE) on the factor graph (FG) representation of $\mathbf{G}_N$. Other approaches rely on faithful estimations of the coordinate channels [21], [22] or Monte Carlo constructions [1], [23].

To select an index set $\mathcal{A}$ for a punctured PC, the channel parameters, *e.g.*, the decision error probabilities or the Bhattacharyya parameters, have to be estimated taking into account the influence of the puncturing pattern. Punctured positions of the codewords are omitted upon transmission and hence result in a reduced code length, but are treated as erasures at decoders designed for the mother code. In a practical system, channel log-likelihood ratios (LLRs) corresponding to these erasures take zero values.

However, these erasures change the characteristics of the coordinate channels $\mathcal{W}_{N,i}$, and totally degrade P of them, rendering P positions in $\mathbf{u}$ useless for information transmission [7], [24]. In this work, we follow [7] and refer to the indices of totally degraded coordinate channels as incapable (information) positions. At such an incapable position i, an SC decoder has to resort to flipping a coin, as we have

$$I(\mathcal{W}_{N,i}) = 0 \tag{6}$$

for the coordinate channel $\mathcal{W}_{N,i}$ modeling the decision about the estimate of u_i.

As a result, modified constructions have to be employed taking the choice of $\mathcal{P}$ into account. For the results presented in this work, we perform DEs to select an index set $\mathcal{A}$ for a given pattern $\mathcal{P}$ as in [7] or [13].

III. Rateless Codes Based on Punctured Polar Codes

We will consider encoding of a fixed number of $K \in \mathbb{N}$ information bits with a rateless code that offers $T \in \mathbb{N}$ different rates. Such a code may hence allow for an HARQ-IR scheme with a maximum number of $T-1$ allowed retransmissions of additional parity information for each block of K information bits. Following, *e.g.*, [15] or [25], in this work we consider a rateless code to be represented by a rate-compatible family of codes obtained via appropriate puncturing. We hence define a rate-compatible family of codes as a set of binary linear codes of increasing length, such that each code has the same dimension and may be obtained by puncturing a longer code of the family.

More formally, such a family

$$\mathfrak{C} := \{\mathcal{C}_t : t \in [T]\} \tag{7}$$

consists of codes $\mathcal{C}_t : \mathbb{F}_2^K \to \mathbb{F}_2^{N_t}$ with lengths $N_t \in \mathbb{N}$ such that $N_t < N_{t+1}$ for each $t \in [T-1]$, and rates $R_t = \frac{K}{N_t}$. Hence, we have $R_t > R_{t+1}$ for each $t \in [T-1]$.

Furthermore, the codes in $\mathfrak{C}$ are nested in the sense that for each consecutive pair of codes $\mathcal{C}_t$, $\mathcal{C}_{t+1} \in \mathfrak{C}$, any codeword $\mathbf{x}_t \in \mathcal{C}_t$ may be obtained from a codeword $\mathbf{x}_{t+1} \in \mathcal{C}_{t+1}$ by puncturing $N_{t+1} - N_t$ positions from $\mathbf{x}_{t+1}$. As a result, in an HARQ-IR scheme, at the t-th transmission, only $N_t - N_{t-1}$ additional bits have to be transmitted, given the receiver was unable to decode after the $(t-1)$-th transmission, usually signaled by a NACK. Hence, in such a scheme, after the t-th transmission, the channel decoder at the receiver attempts to decode based on a noisy observation of a codeword $\mathbf{x}_t$.

To construct a family of rate-compatible codes by puncturing a given (N, K) mother PC $\mathcal{C}$, we construct a sequence of nested puncturing patterns

$$\mathfrak{P} := \{\mathcal{P}_t \subset [N] : t \in [T]\}, \tag{8}$$

that is, a sequence of patterns that satisfy the nesting condition

$$\mathcal{P}_t \supset \mathcal{P}_{t+1} \text{ for each } t \in [T-1]\,. \tag{9}$$

Given such $\mathfrak{P}$, we obtain the codes

$$\mathcal{C}_t = \left\{\mathbf{x}_{\mathcal{P}_t^c} : \mathbf{x} \in \mathcal{C}\right\} \tag{10}$$

for each $t \in [T]$ by puncturing codewords $\mathbf{x} \in \mathcal{C}$ correspondingly, where we write $\mathbf{x}_{\mathcal{P}_t^c} = (x_j : j \in \mathcal{P}_t^c)$. The code $\mathcal{C}_t$ is thus an $(N - |\mathcal{P}_t|, K)$ code, while the length difference between $\mathcal{C}_t$ and $\mathcal{C}_{t+1}$, *i.e.*, the number of additional bits transmitted in the t-th retransmission, is given by $\pi_t := |\mathcal{P}_t \setminus \mathcal{P}_{t+1}|$. We note that not only T, the number of different rates supported by the codes in $\mathfrak{C}$, but also the allocation of length increments π_t are design parameters for the construction of $\mathfrak{C}$.

To accommodate for a simple encoder and decoder design, a fixed index set $\mathcal{A}$, such that $|\mathcal{A}| = K$, for use with all punctured codes $\mathcal{C}_t \in \mathfrak{C}$ is sought. As a result, encoding may be accomplished by standard PC encoders for all codes $\mathcal{C}_t$, and only corresponding puncturing has to be applied in a subsequent step. Furthermore, it facilitates the use of standard decoders, as it enables ad-hoc switching between codes, *i.e.*, switching to a different length and hence rate, without reconfiguring the decoder or even informing it about the code change. In addition to that, it immediately allows for the implementation of HARQ-IR schemes based on $\mathfrak{C}$ without additional encoding, as only additional parity bits may be transmitted, effecting a decreased rate at the decoder which simply takes into account these additional parity bits for decoding.

The following theorem characterizes the complexity of a joint optimization of the index set $\mathcal{A}$ and pattern sequence $\mathfrak{P}$, which may be proved by enumerating the search space. As the number of combinations is already very large for moderate N, such a joint approach is clearly prohibitive.

Theorem 1. *For a fixed T and a given sequence of length increments $\{\pi_t : t \in [T-1]\}$, or equivalently, a fixed sequence of cardinalities for the puncturing patterns in $\mathfrak{P}$, the complexity of constructing $\mathfrak{C}$ by puncturing a mother PC of length $N = 2^n, n \in \mathbb{N}$, by an exhaustive search under the constraint of a common $\mathcal{A}$ encompasses the evaluation of*

$$\binom{N}{K}\binom{N}{|\mathcal{P}_1|}\prod_{t=2}^{T}\binom{|\mathcal{P}_{t-1}|}{|\mathcal{P}_t|} \tag{11}$$

combinations of possible index sets $\mathcal{A} \subset [N]$ and nested puncturing patterns in $\mathfrak{P}$.

A. Construction Example

We illustrate our construction approach for $T = 4$ and consequently construct a rateless code $\mathfrak{C}$ supporting four different rates for a fixed dimension K via length-adaption. We base construction on a mother PC of length $2^8 = 256$, and target a dimension of $K = 96$. Table I gives the parameters used for the codes $\mathcal{C}_t$, where we use $P_t := |\mathcal{P}_t|$. As a result, we have length increments $\pi_t = 32$ for $t \in [3]$. Hence, we form $\mathfrak{C}$ from three punctured versions of the mother PC, and use the unpunctured mother PC as $\mathcal{C}_4$, *i.e.*, $\mathcal{P}_4 = \emptyset$.

TABLE I
CODE PARAMETERS FOR THE CODES $\mathcal{C}_t$ FORMING A RATELESS CODE $\mathfrak{C}$

t	1	2	3	4
N_t	160	192	224	256
P_t	96	64	32	0
R_t	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{3}{8}$

To form the corresponding sequence $\mathfrak{P}$ of puncturing patterns, we use QUP to obtain patterns $\mathcal{P}_t$, which fulfill the nesting constraint given in (9) by construction. Based on these patterns, we then estimate the probabilities $\mathbb{P}[\hat{u}_i \neq u_i]$ via DE. A stylized example of the different punctured versions of a codeword $\mathbf{x} \in \mathcal{C}_4$ is given in Figure 1, while the colors used correspond to the performance results reported in Section IV.

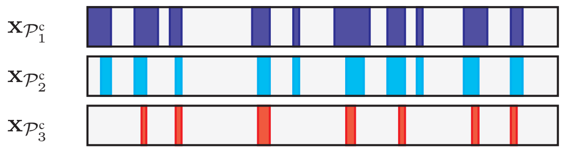

Fig. 1. Stylized examples of the punctured versions of a codeword $\mathbf{x} \in \mathcal{C}_4$ for the codes $\mathcal{C}_t$, $t \in [3]$. Colored areas indicate positions punctured according to the respective puncturing pattern.

IV. EVALUATION AND SIMULATION RESULTS

We evaluate the construction example given in Section III-A under both SC and SCL decoding with a cyclic redundancy check (CRC) as an outer code and a list size of 16. To decode the resulting codes $\mathcal{C}_t$ with an SC decoder, we select $K = |\mathcal{A}| = 96$ positions for inclusion in $\mathcal{A}$, while for CRC-aided SCL decoding, we select $|\mathcal{A}| = 103 = K + 7$ positions, to accommodate for an outer CRC code of length 7 with generator polynomial $g(x) = x^7 + x^3 + 1$ as in [26].

For selecting $\mathcal{A}$, we perform a DE taking into account the largest puncturing pattern $\mathcal{P}_1$ of cardinality $|\mathcal{P}_1| = 96$. We do so to ensure that no code in $\mathfrak{C}$ suffers from catastrophic error propagation under SC decoding due to incapable positions falsely included in $\mathcal{A}$. As the puncturing patterns satisfy the nesting condition, a nesting of the corresponding sets indexing incapable positions is implied. Hence, by assuming worst-case puncturing with $\mathcal{P}_1$ for construction of $\mathcal{A}$, we exclude all incapable positions from $\mathcal{A}$ for all N_t, $t \in [4]$.

Figure 2 reports simulation results to assess the performance of the codes $\mathcal{C}_t \in \mathfrak{C}$ in terms of block error rate (BLER) as well as bit error rate (BER) under both SC and CRC-aided SCL decoding. Under both decoders, with increasing length and thus decreasing rate, performance improves drastically, while all codes show a similar trend. The most heavily punctured code $\mathcal{C}_1$ of length $N_1 = 160$, rate $R_1 = \frac{3}{5}$ only shows acceptable performance under SC decoding in a very high signal-to-noise ratio (SNR) regime (-○-), but benefits more from SCL than the other codes (—●—). In general, the results reported validate the concept of using nested puncturing to construct rateless codes from PCs that offer competitive performance over a wide SNR range for a constant dimension K by adapting the code rate via length adaption.

For evaluating the impact of incapable positions included in the index set, we construct an alternative rateless code $\mathfrak{C}$ using the parameters as given in Table I, but base the index set selection on a DE that only takes into account puncturing pattern $\mathcal{P}_3$ of cardinality 32. Figure 3 gives the resulting BLERs under both SC and CRC-aided SCL decoding as above. We note that this index set selection results in a catastrophic error propagation in the SC decoder for $\mathcal{C}_1$ of length $N_1 = 160$, (-○-), and also SCL decoding does not improve the performance (—●—).

However, when considering $\mathcal{C}_2$ of length $N_2 = 192$, SCL decoding (—●—) overcomes the catastrophic performance degradation observed under SC decoding (-○-). While positions are included in $\mathcal{A}$ that become incapable when puncturing with $\mathcal{P}_2$ (which can be checked easily by, *e.g.*, the algorithm presented in [7]), they do not cluster in consecutive sequences long enough to render SCL decoding useless for the list size of 16 used here. Hence, when list decoding is employed, certain sequences of incapable positions may be included in $\mathcal{A}$ without rendering the punctured code useless, thereby providing additional flexibility for selecting $\mathcal{A}$.

V. CONCLUSION

We present a generic construction of rateless codes by puncturing PCs. By construction, the codes may be decoded with standard PC decoders, and they do not incur additional signaling overhead for ad-hoc rate and length switching. Albeit our approach may be based on arbitrary puncturing methods for PCs, we present an example of our construction based on QUP. Simulations reported for SC and CRC-aided SCL decoding validate our approach, as the resulting codes provide good performance over a wide SNR range.

We note that decreasing the length increments and increasing the number of codes easily allows for rateless constructions offering rates on a finer scale based on the presented approach. In addition to that, we note that SCL decoding offers the potential for relaxed index set selection approaches that consider incapable information positions to a certain extent, depending on the list size.

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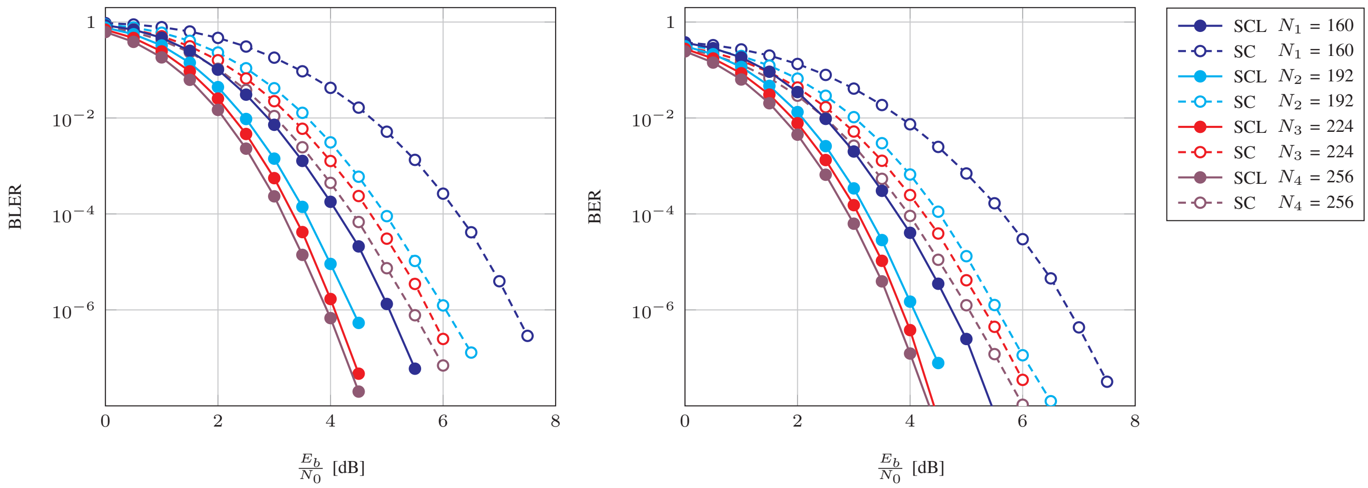

Fig. 2. Performance of the codes $\mathcal{C}_t \in \mathfrak{C}$ under SC and SCL decoding. All codes are of dimension $K = 96$, and are obtained via appropriately puncturing a mother PC of length $N = 256$ using QUP and the parameters given in Table I. The index set $\mathcal{A}$ is selected based on a DE taking into account the pattern $\mathcal{P}_1$ of cardinality $|\mathcal{P}_1| = 96$.

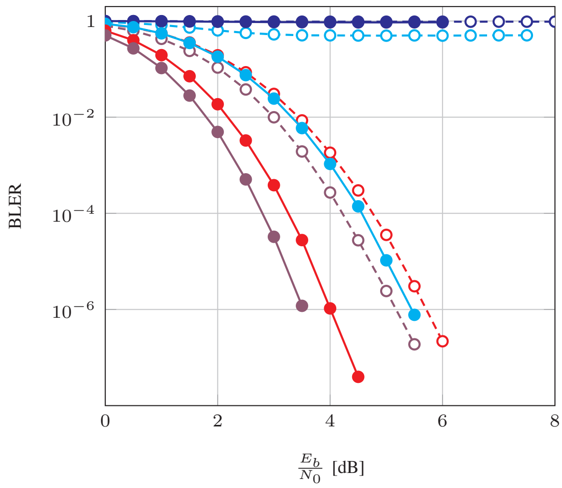

Fig. 3. Catastrophic performance degradation under SC decoding due to incapable positions included in $\mathcal{A}$. The index set $\mathcal{A}$ is selected based on a DE taking into account the puncturing pattern $\mathcal{P}_3$. Please see Figure 2 for legend.

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