The Geometry of the Capacity Region for CDMA Systems With General Power Constraints

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***Abstract*—Access-control strategies and rational pricing in code-division multiple-access (CDMA) systems need exact knowledge of the available transmission capacity and its limiting boundaries. We use the term "capacity region" to specify the set of user-transmission demands that can be supported at the desired quality of service (QoS). In this paper, we investigate the geometrical properties of the capacity region for a fixed number of users in a CDMA radio network under general QoS characteristics and general power constraints. It turns out that, under very mild assumptions, the capacity region is convex, and has an appealing monotonicity property. As a side result, we develop an elementary theory for characterizing the existence of solutions to systems of linear equations with nonnegative elements, analogous to Perron–Frobenius' theory, but bypassing irreducibility.**

***Index Terms*—Capacity region, code-division multiple-access (CDMA), log-convex functions, nonnegative matrices, Perron–Frobenius theory, power assignment, wireless networks.**

I. Introduction and Notation

A central point for understanding the performance of code-division multiple-access (CDMA) in power-controlled environments is the analysis of the intertwining effects of interference from all users. We consider a system with a fixed number n of users, each demanding for a certain quality of service (QoS) characteristic $\gamma_i > 0$. In this work, the set of QoS requirements $(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ that can be supported by an appropriate power assignment is called the "capacity region" of the network. This should be distinguished from the information-theoretic channel capacity, although both concepts aim at a measure of the maximum feasible transmission rate. Related definitions are coined by [3] for joint achievable rates in multiple-access channels and [14] for ad hoc networks (capacity region), moreover by [15] for interference-limited systems (user capacity). In the framework of satellite channels, the authors [10] define the capacity region of arrival-rate vectors in a similar vein.

Comprehending the capacity of CDMA systems has attracted a lot of research activities over the last years. Research along the lines of the present paper has been initiated by [17]. Algorithms, receiver design, and effective bandwidth are included in papers [4], [5], and [15]. Convexity properties of the capacity region and their influence onto the network behavior are investigated in [1] and [7]. In [13], log-convexity of the feasible signal-to-interference ratio (SIR) region is shown, however, power constraints and the effect of how QoS requirements carry over to SIR thresholds in different cases are neglected.

In the present work, we reveal the convexity and monotonicity of the capacity region under general power constraints, thus strongly generalizing previous results and providing a clear unifying theory for the performance of interference-limited multiple-access radio systems. It turns out that convexity is preserved whenever the set of possible power allocations is convex and closed under a simultaneous decrease of power. As a prerequisite, the existence of solutions to systems of linear equations with nonnegative elements is characterized, resembling Perron–Frobenius theory, but avoiding irreducibility.

The present investigations are of high practical relevance for understanding capacity limits, power-control algorithms, and access-control strategies of CDMA networks. Power-control algorithms, e.g., are often designed to maximize QoS over all admissible points in the capacity region $\mathcal{C}_\mathcal{P}$ with power constraints $\mathcal{P}$. This may be formalized by $\min_{\gamma_k \in \mathcal{C}_\mathcal{P}} \sum_{k=1}^n h_k(\gamma_k)$. If h_k are convex functions, and if $\mathcal{C}_\mathcal{P}$ is a convex region, then the powerful theory of convex optimization can be applied, providing a bunch of effective algorithms and powerful characterizations of the existence and uniqueness of solutions.

Access-control strategies rely heavily on the convexity of the capacity region, as is shown in [7]. In the case of congestion, Bayes and minimax policies are suggested, which lead to a fair sharing of the total capacity. Convexity is a prerequisite for the existence and computation of such strategies.

Game theory provides a suitable environment to find rational policies for balancing conflicting interests, in case of congestion. In this framework, the capacity region is interpreted as the set of outcomes in an n-person game. Convexity and a manageable representation are fundamental for determining equilibria strategies.

In the following, let $\boldsymbol{V} = (v_{ij})_{i,j=1}^n$ denote the nonnegative channel gain matrix, with v_{ij} corresponding to the link from transmitter j to receiver i. v_{ij} is used as a multiplicative constant describing the power decrease on the respective path. We, furthermore, assume that j's decoding receiver is labeled j, such that v_{jj} is the attenuation on the path linking j to the network.

The case wherein many transmitters are served by the same receiver, as what occurs for a CDMA uplink with many mobiles and only a few base stations, is also covered. If the locations and physical properties of receivers i and k coincide, then the corresponding lines i, k of $\boldsymbol{V}$ are identical. It should be mentioned that an analogous model, also including activity and

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orthogonality factors, can be applied to the downlink. This form of duality is investigated in [2] and [8].

Irreducibility is a rather natural property in this context. Matrix $\boldsymbol{V}$ is called irreducible if, for any i, j, there is a finite chain $i_m \in \{1,\ldots,n\}$, $m = 0,1,\ldots,k$, $i_0 = i$, $i_k = j$, such that $v_{i_m,i_{m+1}} > 0$. However, it turns out that the assumption of irreducibility can be omitted for characterizing solutions of positive power. Besides the mathematical elegance, this makes the model more widely applicable.

II. Power Assignment and the Capacity Region

For every power allocation $\boldsymbol{p} = (p_1,\ldots,p_n)' > \boldsymbol{0}$, let

$$\mathrm{SIR}_k(\boldsymbol{p}) = \frac{v_{kk}p_k}{\sum_{l\neq k} v_{kl}p_l + C_k\sigma^2}, \qquad k = 1,\ldots,n$$

where $\sigma^2 > 0$, $C_k > 0$, $v_{kk} > 0$ for all k, and $v_{kl} \geq 0$ for all k, l.

Let $\psi : (0,\infty) \to (0,\infty)$ be strictly increasing or strictly decreasing. ψ is a monotonic transformation of the demands $\gamma_1,\ldots,\gamma_n$ of n users corresponding to a certain QoS characteristic. In this paper, we are particularly interested in log-convex functions ψ. This attribute entails the convexity of the capacity region. Some examples of monotonic, log-convex functions ψ are

$$\psi(\gamma) = \mathrm{e}^{\gamma}, \quad \text{(direct log-SIR requirements)}$$

$$\psi(\gamma) = \frac{1}{\gamma}, \quad \text{(effective spreading-gain requirements)}$$

$$\psi(\gamma) = \frac{\mathrm{e}^{\gamma}}{(1-\mathrm{e}^{\gamma})}, \quad \text{(effective bandwidth requirements).}$$

The unconstrained capacity region is now defined as the set of demand vectors, such that there exists a power assignment $\boldsymbol{p}$ satisfying all SIR requirements, i.e.

$$\mathcal{C} = \{\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_n)' > \boldsymbol{0} \mid \exists \boldsymbol{p} > \boldsymbol{0} : \mathrm{SIR}_k(\boldsymbol{p}) \geq \psi(\gamma_k)\ \forall k\}.$$

To cover the case of limited power, we introduce the set $\mathcal{P} \subset (0,\infty)^n$ of admissible power allocations. Correspondingly, we define the constrained capacity region by

$$\mathcal{C}_{\mathcal{P}} = \{\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_n)' > \boldsymbol{0} \mid \exists \boldsymbol{p} \in \mathcal{P} : \mathrm{SIR}_k(\boldsymbol{p}) \geq \psi(\gamma_k)\ \forall k\}.$$

We assume throughout that $\mathcal{P}$ is closed under simultaneous decrease of power, that is

$$\text{if } \boldsymbol{p} \in \mathcal{P} \text{ and } \boldsymbol{0} < \boldsymbol{q} \leq \boldsymbol{p}, \text{ then } \boldsymbol{q} \in \mathcal{P}. \tag{1}$$

Write $\boldsymbol{B} = (b_{kl})_{k,l=1}^n$, where

$$b_{kl} = \begin{cases} \frac{v_{kl}}{v_{kk}}, & k \neq l \\ 0, & k = l \end{cases}$$

and $\boldsymbol{\tau} = (\tau_1,\ldots,\tau_n)'$, where $\tau_k = C_k\sigma^2/v_{kk}$. Then, for every $\boldsymbol{p} > \boldsymbol{0}$

$$\mathrm{SIR}_k(\boldsymbol{p}) = \frac{p_k}{\sum_{l=1}^n b_{kl}p_l + \tau_k}, \qquad k = 1,\ldots,n.$$

Moreover, $\mathrm{SIR}_k(\boldsymbol{p}) \geq \psi(\gamma_k)$ for all k, if and only if

$$[\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}]\,\boldsymbol{p} \geq \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{\tau}$$

where $\boldsymbol{\Psi}(\boldsymbol{\gamma}) = \mathrm{diag}(\psi(\gamma_1),\ldots,\psi(\gamma_n))$. It now follows from Lemma 1 d) in the Appendix that

$$\mathcal{C} = \{\boldsymbol{\gamma} > \boldsymbol{0} \mid \exists \boldsymbol{p} > \boldsymbol{0} : [\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}]\,\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{\tau}\}$$

and in view of condition (1)

$$\mathcal{C}_{\mathcal{P}} = \{\boldsymbol{\gamma} > \boldsymbol{0} \mid \exists \boldsymbol{p} \in \mathcal{P} : [\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}]\,\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{\tau}\}.$$

Let $\rho(\boldsymbol{A})$ denote the spectral radius of a matrix $\boldsymbol{A}$. By Lemma 1 a) and b), for any given $\boldsymbol{\gamma} > \boldsymbol{0}$, the equation $[\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}]\boldsymbol{p} = \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{\tau}$ has a positive solution $\boldsymbol{p}$, if, and only if, $\rho(\boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}) < 1$, and in that case, the solution is unique. Denote it by $\boldsymbol{\pi}(\boldsymbol{\gamma}) = (\pi_1(\boldsymbol{\gamma}),\ldots,\pi_n(\boldsymbol{\gamma}))'$. Thus

$$\boldsymbol{\pi}(\boldsymbol{\gamma}) = [\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}]^{-1}\,\boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{\tau} \in (0,\infty)^n \qquad \text{for all } \gamma \in \mathcal{C}$$

and

$$\mathcal{C} = \{\boldsymbol{\gamma} > \boldsymbol{0} \mid \rho(\boldsymbol{\Psi}(\boldsymbol{\gamma})\boldsymbol{B}) < 1\}, \quad \mathcal{C}_{\mathcal{P}} = \{\boldsymbol{\gamma} \in \mathcal{C} \mid \boldsymbol{\pi}(\boldsymbol{\gamma}) \in \mathcal{P}\}.$$

III. Monotonicity and Convexity of Power Allocations

Monotonicity of the capacity region is, inter alia, important for the optimum base-station assignment. The authors [9] give a characterization of the set of feasible values in the case of unlimited power by its boundary surfaces, and show that power uniformly rises whenever some QoS demand is increased. The following is a general monotonicity property, simultaneously of the set of demand vectors $\mathcal{C}$ and the corresponding power allocations.

Theorem 1: Suppose that ψ is strictly increasing. Let $\boldsymbol{\gamma}^{(1)} \in \mathcal{C}$ and $\boldsymbol{0} < \boldsymbol{\gamma}^{(0)} \leq \boldsymbol{\gamma}^{(1)}$, $\boldsymbol{\gamma}^{(0)} \neq \boldsymbol{\gamma}^{(1)}$.

a) It holds that $\boldsymbol{\gamma}^{(0)} \in \mathcal{C}$, $\boldsymbol{\pi}(\boldsymbol{\gamma}^{(0)}) \leq \boldsymbol{\pi}(\boldsymbol{\gamma}^{(1)})$, and if $\gamma_k^{(0)} \neq \gamma_k^{(1)}$, then $\pi_k(\boldsymbol{\gamma}^{(0)}) < \pi_k(\boldsymbol{\gamma}^{(1)})$.
b) If $\boldsymbol{B}$ is irreducible, then $\boldsymbol{\pi}(\boldsymbol{\gamma}^{(0)}) < \boldsymbol{\pi}(\boldsymbol{\gamma}^{(1)})$.

The same is true when ψ is strictly decreasing and $\boldsymbol{\gamma}^{(0)} \geq \boldsymbol{\gamma}^{(1)} \in \mathcal{C}$, $\boldsymbol{\gamma}^{(0)} \neq \boldsymbol{\gamma}^{(1)}$.

Proof: We consider only the case where ψ is strictly increasing.

a) As $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{B} \leq \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{B}$, by the monotonicity of the spectral radius (see, e.g., [6, Corollary 8.1.19]), we conclude that $\rho(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{B}) \leq \rho(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{B}) < 1$, so that $\boldsymbol{\gamma}^{(0)} \in \mathcal{C}$. Hence, for $i = 0, 1$

$$\boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(i)}\right) = \left[\boldsymbol{I} - \boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(i)}\right)\boldsymbol{B}\right]^{-1}\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(i)}\right)\boldsymbol{\tau} = \boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(i)}\right)\boldsymbol{\tau} + \left\{\sum_{\nu=1}^{\infty}\left[\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(i)}\right)\boldsymbol{B}\right]^{\nu}\right\}\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(i)}\right)\boldsymbol{\tau}.$$

Since $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{\tau} \leq \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{\tau}$ and $[\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{B}]^{\nu} \leq [\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{B}]^{\nu}$ for all ν, it follows that $\boldsymbol{\pi}(\boldsymbol{\gamma}^{(0)}) \leq \boldsymbol{\pi}(\boldsymbol{\gamma}^{(1)})$. If $\gamma_k^{(0)} < \gamma_k^{(1)}$, then $\{\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{\tau}\}_k < \{\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{\tau}\}_k$, and it then follows that $\pi_k(\boldsymbol{\gamma}^{(0)}) < \pi_k(\boldsymbol{\gamma}^{(1)})$.

b) If $\boldsymbol{B}$ is irreducible, so is $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{B}$. Thus, for every pair $(k,l) \in \{1,\ldots,n\}^2$, there exists $\nu^* = \nu^*(k,l) \geq 0$, such that $\{[\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{B}]^{\nu^*}\}_{kl} > 0$. The matrix $\sum_{\nu=0}^{\infty}[\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{B}]^{\nu}$ is, therefore, strictly positive. Consequently, as $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{\tau} \leq \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{\tau}$ and $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(0)})\boldsymbol{\tau} \neq \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(1)})\boldsymbol{\tau}$

$$\boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(0)}\right) = \left\{\sum_{\nu=0}^{\infty}\left[\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(0)}\right)\boldsymbol{B}\right]^{\nu}\right\}\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(0)}\right)\boldsymbol{\tau} < \left\{\sum_{\nu=0}^{\infty}\left[\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(0)}\right)\boldsymbol{B}\right]^{\nu}\right\}\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(1)}\right)\boldsymbol{\tau} \leq \boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(1)}\right).$$

■

The next theorem shows that the individual power allocations $\pi_k(\boldsymbol{\gamma})$ follow a convex curve when $\boldsymbol{\gamma}$ moves along a straight line through the capacity region.

Theorem 2: Suppose that ψ is log-convex. Let $\boldsymbol{\gamma}^{(0)}, \boldsymbol{\gamma}^{(1)} \in \mathcal{C}$, $\boldsymbol{\gamma}^{(0)} \neq \boldsymbol{\gamma}^{(1)}$. Set $\boldsymbol{\gamma}^{(\lambda)} = \lambda\boldsymbol{\gamma}^{(1)} + (1-\lambda)\boldsymbol{\gamma}^{(0)}$ for $0 \leq \lambda \leq 1$.

a) For every $k = 1,\ldots,n$, $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is a convex function of $\lambda \in [0,1]$. If $\gamma_k^{(0)} \neq \gamma_k^{(1)}$, then $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is strictly convex.

b) If $\boldsymbol{B}$ is irreducible, then for every k, $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is strictly convex.

Proof:

a) Define $\boldsymbol{g} : [0,1] \to (0,\infty)^n$ and $\boldsymbol{h} : [0,1] \to [0,\infty]^n$ by

$$\boldsymbol{g}(\lambda) = \boldsymbol{\Psi}\left(\gamma^{(\lambda)}\right)\boldsymbol{\tau},$$

$$\boldsymbol{h}(\lambda) = \left\{\sum_{\nu=1}^{\infty}\left[\boldsymbol{\Psi}\left(\gamma^{(\lambda)}\right)\boldsymbol{B}\right]^{\nu}\right\}\boldsymbol{\Psi}\left(\gamma^{(\lambda)}\right)\boldsymbol{\tau}, \quad 0 \leq \lambda \leq 1.$$

The series converges if, and only if, $\rho(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(\lambda)})\boldsymbol{B}) < 1$. Note that the series converges for $\lambda = 0$ and $\lambda = 1$. For all $\lambda \in [0,1]$ with $\rho(\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(\lambda)})\boldsymbol{B}) < 1$

$$\begin{aligned}\boldsymbol{g}(\lambda) + \boldsymbol{h}(\lambda) &= \left\{\sum_{\nu=0}^{\infty}\left[\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(\lambda)}\right)\boldsymbol{B}\right]^{\nu}\right\}\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(\lambda)}\right)\boldsymbol{\tau} \\ &= \left[\boldsymbol{I} - \boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(\lambda)}\right)\boldsymbol{B}\right]^{-1}\boldsymbol{\Psi}\left(\boldsymbol{\gamma}^{(\lambda)}\right)\boldsymbol{\tau} \\ &= \boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(\lambda)}\right). \end{aligned} \tag{2}$$

Let $\mathcal{L}$ denote the class of log-convex functions on [0, 1] augmented by the zero function. Since ψ is log-convex, every entry of $\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(\lambda)})$ belongs to $\mathcal{L}$. Following an approach of [2], we now take advantage of the fact that the class $\mathcal{L}$ is closed under the following operations, see [11, p. 19]. If $\alpha \geq 0$ and $f_1, f_2 \in \mathcal{L}$, then $\alpha f_1, f_1 + f_2, f_1 \cdot f_2 \in \mathcal{L}$. Moreover, if $(f_\nu)_{\nu=1}^{\infty} \subset \mathcal{L}$ and $\sum_{\nu=1}^{\infty} f_\nu(\lambda) < \infty$ for $\lambda \in \{0,1\}$, then $\sum_{\nu=1}^{\infty} f_\nu(\lambda) < \infty$ for all $\lambda \in [0,1]$ and $\sum_{\nu=1}^{\infty} f_\nu \in \mathcal{L}$. Using these properties, one obtains that for every $k = 1,\ldots,n$, $h_k(\lambda) < \infty$ for all $0 \leq \lambda \leq 1$, and $g_k, h_k \in \mathcal{L}$. In particular, (2) holds for all $0 \leq \lambda \leq 1$, and so $\pi_k(\boldsymbol{\gamma}^{(\lambda)}) = g_k(\lambda) + h_k(\lambda) \in \mathcal{L}$, too. As log-convex functions are convex, $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is therefore convex for every k.

If $\gamma_k^{(0)} \neq \gamma_k^{(1)}$, then, since ψ is strictly monotonic, $g_k(\lambda) = \tau_k\psi(\gamma_k^{(0)} + \lambda(\gamma_k^{(1)} - \gamma_k^{(0)}))$ is strictly monotonic as well. Thus, by Lemma 2 a), g_k is strictly convex. As h_k is convex, it follows that $\pi_k(\boldsymbol{\gamma}^{(\lambda)}) = g_k(\lambda) + h_k(\lambda)$ is strictly convex.

b) Suppose that $\boldsymbol{B}$ is irreducible. For an indirect proof, assume that there exists some $k \in \{1,\ldots,n\}$, such that $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is not strictly convex. According to a), this implies that $\gamma_k^{(0)} = \gamma_k^{(1)}$, so that $\{\boldsymbol{\Psi}(\boldsymbol{\gamma}^{(\lambda)})\}_{kk} = \psi(\gamma_k^{(0)})$ for all $0 \leq \lambda \leq 1$. Moreover, since $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is log-convex, it follows from Lemma 2 a) that $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ is constant on some nondegenerate interval $\Lambda \subset [0,1]$. The kth row of the equation $[\boldsymbol{I} - \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(\lambda)})\boldsymbol{B}]\boldsymbol{\pi}(\boldsymbol{\gamma}^{(\lambda)}) = \boldsymbol{\Psi}(\boldsymbol{\gamma}^{(\lambda)})\boldsymbol{\tau}$ yields that

$$\pi_k\left(\boldsymbol{\gamma}^{(\lambda)}\right) - \psi\left(\gamma_k^{(0)}\right)\sum_{l=1}^{n} b_{kl}\pi_l\left(\boldsymbol{\gamma}^{(\lambda)}\right) = \psi\left(\gamma_k^{(0)}\right)\tau_k \quad \text{for all } 0 \leq \lambda \leq 1.$$

Hence, $\sum_{l=1}^{n} b_{kl}\pi_l(\boldsymbol{\gamma}^{(\lambda)})$ is constant on Λ. Since each of the functions $\pi_l(\boldsymbol{\gamma}^{(\lambda)})$ is log-convex, it follows from Lemma 2 b) that $\pi_l(\boldsymbol{\gamma}^{(\lambda)})$ is constant on Λ, for every l with $b_{kl} > 0$. It follows by induction that $\pi_m(\boldsymbol{\gamma}^{(\lambda)})$ is not strictly convex on [0, 1] for all m, for which there exist $k_1,\ldots,k_\mu \in \{1,\ldots,n\}$, such that $k_1 = k$, $k_\mu = m$, and $b_{k_\nu,k_{\nu+1}} > 0$ for $\nu = 1,\ldots,\mu-1$. However, if $\boldsymbol{B}$ is irreducible, such a sequence exists for every $m \in \{1,\ldots,n\}$, so that none of the functions $\pi_m(\boldsymbol{\gamma}^{(\lambda)})$ is strictly convex. This contradicts a). ■

The proof of Theorem 2 shows the slightly stronger assertion that the $\pi_k(\boldsymbol{\gamma}^{(\lambda)})$ are even log-convex functions which, under the conditions stated, are not constant on any subinterval.

IV. Geometry of the Capacity Region

We now state the central results on the geometry of the capacity region. Both results are obtained as corollaries to Theorem 1 and Theorem 2 a), respectively.

Corollary 1: Suppose that the set $\mathcal{P}$ of admissible power allocations satisfies (1). If ψ is strictly increasing and $\boldsymbol{\gamma}^{(1)} \in \mathcal{C}_{\mathcal{P}}$, then $\boldsymbol{\gamma} \in \mathcal{C}_{\mathcal{P}}$ for all $0 < \boldsymbol{\gamma} \leq \boldsymbol{\gamma}^{(1)}$. If ψ is strictly decreasing and $\boldsymbol{\gamma}^{(1)} \in \mathcal{C}_{\mathcal{P}}$, then $\boldsymbol{\gamma} \in \mathcal{C}_{\mathcal{P}}$ for all $\boldsymbol{\gamma} \geq \boldsymbol{\gamma}^{(1)}$.

Corollary 1 shows the monotonicity of the capacity region $\mathcal{C}_{\mathcal{P}}$. For any fixed demand vector $\boldsymbol{\gamma}^{(1)} \in \mathcal{C}_{\mathcal{P}}$, all vectors componentwise less (greater) than or equal to $\boldsymbol{\gamma}^{(1)}$ are feasible as well, provided ψ is strictly increasing (decreasing). Hence, transmission remains feasible if $\boldsymbol{\gamma}$ is uniformly decreased (increased). Corollary 1 points out which $\boldsymbol{\gamma}$ values might lead to critical load situations. It, furthermore, describes the geometrical shape of $\mathcal{C}_{\mathcal{P}}$.

Convexity follows from the convexity of $\mathcal{P}$ by applying Theorem 2 a).

Corollary 2: Suppose that ψ is log-convex. Suppose further that the set $\mathcal{P}$ of admissible power allocations is convex and satisfies (1). Then, the constrained capacity region $\mathcal{C}_{\mathcal{P}}$ is a convex set.

Proof: Let $\boldsymbol{\gamma}^{(0)}, \boldsymbol{\gamma}^{(1)} \in \mathcal{C}_{\mathcal{P}}$. Thus, $\boldsymbol{\pi}(\boldsymbol{\gamma}^{(0)}), \boldsymbol{\pi}(\boldsymbol{\gamma}^{(1)}) \in \mathcal{P}$. Write $\boldsymbol{\gamma}^{(\lambda)} = \lambda\boldsymbol{\gamma}^{(1)} + (1-\lambda)\boldsymbol{\gamma}^{(0)}$. Theorem 2 a) and the assumption that $\mathcal{P}$ be convex yield that

$$\boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(\lambda)}\right) \leq \lambda\boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(1)}\right) + (1-\lambda)\boldsymbol{\pi}\left(\boldsymbol{\gamma}^{(0)}\right) \in \mathcal{P} \quad \text{for all } 0 \leq \lambda \leq 1.$$

Hence, by (1), $\boldsymbol{\pi}(\boldsymbol{\gamma}^{(\lambda)}) \in \mathcal{P}$, and, therefore, $\boldsymbol{\gamma}^{(\lambda)} \in \mathcal{C}_{\mathcal{P}}$, for all $0 \leq \lambda \leq 1$. ■

We conclude with some remarks concerning the boundary of the capacity region, which is important for the uniqueness of solutions to optimal control and access problems (see [7]). Let ψ and $\mathcal{P}$ satisfy the assumptions of Theorem 2. Then, the surface $\widehat{\mathcal{C}}_{\mathcal{P}} = \{\boldsymbol{\gamma} \in \mathcal{C} | \boldsymbol{\pi}(\boldsymbol{\gamma}) \in \partial\mathcal{P}\}$ contains no line segments, provided that the matrix $\boldsymbol{B}$ is irreducible or the set $\mathcal{P}$ satisfies the stronger condition that

$$\text{if } \boldsymbol{p} \in \overline{\mathcal{P}} \text{ and } \boldsymbol{0} < \boldsymbol{q} \leq \boldsymbol{p},\ \boldsymbol{q} \neq \boldsymbol{p}, \text{ then } \boldsymbol{q} \in \text{int}\mathcal{P}. \quad (3)$$

Condition (3) is satisfied if $\mathcal{P}$ is of the form $\mathcal{P} = \{\boldsymbol{p} > \boldsymbol{0} \mid \sum_{k=1}^{n} p_k \leq p_{\max}\}$, that is, if the total power is restricted. Capacity regions corresponding to this special type of constraints have been studied in [1]. In turn, condition (3) is violated for power regions of the form $\mathcal{P} = \{\boldsymbol{p} > \boldsymbol{0} \mid p_k \leq \hat{p}_k \text{ for all } k\}$, that is, if the power of each mobile is restricted separately. The corresponding capacity regions and optimal power allocations are investigated in [7] for the case where $\psi(x) = 1/x$.

APPENDIX

The following lemma is concerned with solutions of the equation

$$[\boldsymbol{I} - \boldsymbol{A}]\boldsymbol{x} = \boldsymbol{c} \quad (4)$$

when $\boldsymbol{A}$ is a nonnegative, but not necessarily irreducible, matrix. It is an elementary analog of [12, Th. 2.1] and does not rest on the theory of Perron and Frobenius. Irreducibility is also avoided in [16, Th. 1]. In this work, the assumption of irreducibility would prevent application of the model in the multiuser receiver framework.

Lemma 1: Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be nonnegative.

a) If there are $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{c} > \boldsymbol{0}$ satisfying (4), then $\rho(\boldsymbol{A}) < 1$.
b) If $\rho(\boldsymbol{A}) < 1$, then $\boldsymbol{I} - \boldsymbol{A}$ is nonsingular, and for every $\boldsymbol{c} > \boldsymbol{0}$, the unique solution $\boldsymbol{x} \in \mathbb{R}^n$ of (4) is positive.
c) If $\rho(\boldsymbol{A}) < 1$, then for every $\boldsymbol{c} \geq \boldsymbol{0}$, the unique solution $\boldsymbol{x} \in \mathbb{R}^n$ of (4) is nonnegative.
d) If $\boldsymbol{c} > \boldsymbol{0}$, and there exists $\boldsymbol{y} > \boldsymbol{0}$, such that $[\boldsymbol{I} - \boldsymbol{A}]\boldsymbol{y} \geq \boldsymbol{c}$, then (4) has a unique solution $\boldsymbol{x}$ and $\boldsymbol{0} < \boldsymbol{x} \leq \boldsymbol{y}$.

Proof:

a) Suppose $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{c} > \boldsymbol{0}$ satisfy (4). A well-known bound for the spectral radius of a nonnegative matrix [6, Theorem 8.1.26, p. 493] yields that, for some i, $\rho(\boldsymbol{A}) \leq x_i^{-1} \sum_{j=1}^{n} a_{ij}x_j = 1 - c_i/x_i < 1$.
b) and c) This is obvious by writing $[\boldsymbol{I} - \boldsymbol{A}]^{-1}$ as a geometric von Neumann series.
d) If $\boldsymbol{y} > \boldsymbol{0}$ and $\boldsymbol{d} := [\boldsymbol{I} - \boldsymbol{A}]\boldsymbol{y} \geq \boldsymbol{c} > \boldsymbol{0}$, then, by a), $\rho(\boldsymbol{A}) < 1$. Thus, by b), there is a unique $\boldsymbol{x} > \boldsymbol{0}$, such that $[\boldsymbol{I} - \boldsymbol{A}]\boldsymbol{x} = \boldsymbol{c}$. Since $[\boldsymbol{I} - \boldsymbol{A}](\boldsymbol{y} - \boldsymbol{x}) = \boldsymbol{d} - \boldsymbol{c} \geq \boldsymbol{0}$, it follows from c) that $\boldsymbol{y} - \boldsymbol{x} \geq \boldsymbol{0}$. ■

The next lemma provides two simple properties of log-convex functions.

Lemma 2:

a) If $f : [a, b] \to (0, \infty)$ is log-convex, and there is no subinterval of $[a, b]$, where f is constant, then f is strictly convex.
b) If $f,\ g : [a, b] \to (0, \infty)$ are log-convex, and $f + g$ is constant on $[a, b]$, then both f and g are constant on $[a, b]$.

Proof:

a) Assume f is log-convex, but not strictly convex. Thus, f is convex, but not strictly so, and it follows that f is affine on some interval. Since f is log-convex, f must be constant on that interval.
b) If f and g are log-convex, they are convex, and since $f = \text{const} - g$, f and g are both convex and concave, that is, affine; but affine log-convex functions are constant. ■

Lemma 2 b) does not carry over to products. That is, there exist log-convex functions f and g, such that fg is constant, but neither f, nor g, is constant. For example, $f(x) = \mathrm{e}^{x}$ and $g(x) = \mathrm{e}^{-x}$.

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