Gaussian Random Fields as a Model for Spatially Correlated Log-Normal Fading

Daniel Catrein, Rudolf Mathar
Institute for Theoretical Information Technology
RWTH Aachen University
D-52056 Aachen, Germany
{catrein, mathar}@ti.rwth-aachen.de

***Abstract*— Slow fading or shadowing on a wireless channel is commonly modeled by stochastically independent, log-normally distributed random variables. However, as slow fading is caused by buildings and large size obstacles, spatial correlations occur. In this paper, Gaussian random fields are used as a model for correlated slow fading in urban environments. An exponential correlation function is employed. The according parameters are estimated from path gain predictions by an accurate ray-optical propagation algorithm, named CORLA. Furthermore, a multidimensional model is suggested to describe correlated shadowing of the path gains from different base stations to a single receiver.**

I. Introduction

Wireless channel modeling is an important prerequisite for the design of mobile communication systems. In this paper, we focus on large scale fading, or shadowing, induced by the relatively slow motion of mobiles and according attenuation, reflection and diffraction of radio waves at buildings and other obstacles in urban environments. The log-normal distribution, well motivated by the central limit theorem on a dB scale, is often used to model the path gain fluctuations. Most simulations use independent log-normal random variables as a model for slow fading over time. This comes at low computational complexity, however, neglects correlations between samples at close quarters. Taking the correlation structure into account usually means high computational costs. Hence, an important goal are computationally effective models of the spatial correlation structure of slow fading in urban environments.

One of the first publications considering correlations for shadow fading is [1]. After performing measurements, the authors propose to model shadow fading as a stochastic process with an exponential type correlation function and Gaussian marginal distributions. The authors [2] discuss a statistical model to generate shadow-fading which uses a precomputed fading map. Correlations are introduced by a Gaussian random field and parameters are computed from local measurements. Further studies of correlated radio channels, well supported by measurements, are presented in [3], [4], [5]. The correlation between measurement points is described as a function of their distance.

A general problem with measurements is that they are commonly taken along one-dimensional routes. A measurement campaign aiming at a whole map of channel gains for an area of several square kilometers around a receiver with high resolution is certainly impractical. However, for the analysis of the spatial correlation structure this data is simply needed. Shadow fading is caused by buildings and other large obstacles. Modern three-dimensional field strength prediction tools include such obstacles in their computations. Hence, we suggest the use of a well calibrated field strength prediction tool to generate the input data for parameter estimation of a generic statistical spatial fading model.

In this work, we focus on Gaussian random fields as a model for spatially correlated fading. Section II introduces some basic notations, the path loss model and some parameter estimators. In Section III, parameter estimation of the generic stochastic model from a radio wave propagation prediction tool is described in detail. Section IV summarizes some basic definitions and important facts about Gaussian random fields. Furthermore, we discuss if such fields are appropriate for modeling spatially correlated fading. In Section V we introduce a correlation model for the path gain of different base stations at a single receiver. Section VI concludes the paper with a brief outlook on open questions.

II. Path-loss Model and Parameter Estimation

Assume a receiver at location $\boldsymbol{x} \in \mathbb{R}^2$ and the transmitting base station located at the origin. Neglecting fast fading and assuming transmit and receive antenna gains of G_t and G_r, respectively, the path loss in dB from the base station to the receiver is modeled by

$$L_{\mathrm{dB}}(\boldsymbol{x}) = -20\log\frac{\lambda}{4\pi} - 10\log(G_tG_r) - \alpha 10\log\|\boldsymbol{x}\| + Y_{\mathrm{dB}}(\boldsymbol{x}). \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm or distance from the origin, α the path loss exponent and $Y_{\mathrm{dB}}(\boldsymbol{x})$ additive log-normal fading. We also write $L_{\mathrm{dB}}(d)$, if $d = \|\boldsymbol{x}\|$ is the distance of location $\boldsymbol{x}$ from the transmitter at the origin, and $d^{\mathrm{dB}} = 10\log d$. The random variables $Y_{\mathrm{dB}}(\boldsymbol{x})$ are assumed to be spatially correlated for different $\boldsymbol{x}$. For fixed $\boldsymbol{x}$ the marginal distribution of $Y_{\mathrm{dB}}(\boldsymbol{x})$ is supposed to be $\mathrm{N}(0,\sigma^2)$, i.e., Gaussian with expectation 0 and variance σ^2, independent of $\boldsymbol{x}$.

Let $y_{d_1},\dots,y_{d_n}$ be a set of n independent observations of $L_{\mathrm{dB}}(d_1),\dots,L_{\mathrm{dB}}(d_n)$ at locations $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$, and let $c = -20\log\frac{\lambda}{4\pi} - 10\log(G_tG_r)$ denote the constant term in (1).

978-1-4244-2603-4/08/$25.00 © 2008 IEEE

The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\sigma}$ of α and σ, respectively, are given as

$$\hat{\alpha} = -\frac{\sum_{i=1}^{n} d_i^{\mathrm{dB}}(y_{d_i} - c)}{\sum_{i=1}^{n}(d_i^{\mathrm{dB}})^2}, \tag{2}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_{d_i} - c + \hat{\alpha}\, d_i^{\mathrm{dB}})^2}. \tag{3}$$

This is easily derived by differentiating the log-likelihood function w.r.t. α and σ, and determining zeros of the gradient.

Measurements in close vicinity are of course correlated. We assume that stochastically independent samples may be drawn by observations which are sufficiently far apart.

The above estimators are used in the next section to compute the path-loss exponent α and hence subtract the distance dependent part of path loss from the observations.

III. Parameter Estimation by Ray Launching

The main purpose of this work is to find a generic stochastic model for spatially correlated shadowing. Estimating the parameters is a crucial point for adapting the model to realistic scenarios. This may be done on the basis of measurements, a way which is, however, tedious, time intensive and requires expensive equipment. In the present approach, we instead employ the results of an accurate radio wave propagation tool for estimation purposes. Precise path gain predictions are obtained from the tool CORLA [6]. The predictions have been validated against measurements carried out by former Mannesmann Mobilfunk GmbH, Germany (today: Vodafone D2 GmbH, Germany) within the COST 231 project [7].

Figure 1(a) shows a CORLA prediction with a resolution of 5 meters for the center of Munich. The measurements displayed in Figure 1(b) have been collected on a certain winded track araound the transmitter in Figure 1(a). The predicted values are in excellent accordance with the measured field strength in Figure 1(b). An extensive analysis of the prediction accuracy is given in [6]. In summary, estimating the parameters of a generic model from concrete path gain predictions is well-grounded and leads to valuable results.

The path loss exponent α and the shadowing variance σ of model (1) are now estimated by (2) and (3). The following values are obtained

$$\hat{\alpha} = 3.57 \text{ and } \hat{\sigma} = 7.14,$$

in accordance with typical values resulting from measurements in urban scenarios. Subtracting the distance dependent part of (1) from the predictions by using $\hat{\alpha}$ leads to Figure 1(c). This Figure hence represents a shadowing map. Red colors indicate regions where the path gain is higher than the purely distance dependent part. Green colors represent regions with a low path gain due to shadowing, and white areas correspond to regions where the path gain is in accordance with the distance dependent value.

Figure 1(d) contains a histogram of the slow fading values occurring in Figure 1(c)). It demonstrates that the values compare acceptably with the density of a Gaussian $\mathrm{N}(0, \hat{\sigma}^2)$ distribution, depicted in the same plot and assumed in model (1). Deviations are mainly caused by discretization and rounding errors since CORLA only provides dB values on a discrete scale.

In the following section we aim at adapting a generic stochastic field model to the typical statistical properties of spatial radio wave propagation in urban scenarios.

IV. Random Fields as a Model for Correlated Shadowing

A random field is a stochastic process $\{X(\boldsymbol{x})\}$ where the index $\boldsymbol{x}$ ranges over a multidimensional space, e.g., $\boldsymbol{x} \in \mathbb{R}^n, n \geq 2$, or a subset $\mathcal{A}$ hereof. A comprehensive survey of the theory and applications of random fields is provided in [8]. Gaussian random fields as models for shadow fading are also investigated in the work [2], however for a different type of spatial correlation and without considering parameter estimation.

Susequently some basic definitions are summarized in short. We assume that occuring expectations are all well defined.

The *covariance function* of a random field is defined as

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathrm{E}\big(X(\boldsymbol{x}_1)X(\boldsymbol{x}_2)\big) - \mathrm{E}\big(X(\boldsymbol{x}_1)\big)\ \mathrm{E}\big(X(\boldsymbol{x}_2)\big),$$

the *correlation function* as

$$R(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{C(\boldsymbol{x}_1, \boldsymbol{x}_2)}{\sqrt{\mathrm{V}\big(X(\boldsymbol{x}_1)\big)}\ \sqrt{\mathrm{V}\big(X(\boldsymbol{x}_2)\big)}}.$$

A function $R(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the correlation function of some random field if and only if it is symmetric and nonnegative definite.

Generalizing the concept of stationary stochastic processes to random fields leads to the concept of homogeneous random fields. A random field is called *weak-sense homogeneous* if

$$\mathrm{E}\big(X(\boldsymbol{x})\big) = \mu \quad \text{and} \quad R(\boldsymbol{x}_1, \boldsymbol{x}_2) = R(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$

for all $\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2$. A weak-sense homogeneous random field is called *weak-sense isotropic* if

$$R(\boldsymbol{x}_1, \boldsymbol{x}_2) = R(\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|)$$

for all $\boldsymbol{x}_1, \boldsymbol{x}_2$. Nonnegative definiteness is also a necessary and sufficient condition for a function R to be the the correlation function of an isotropic random field.

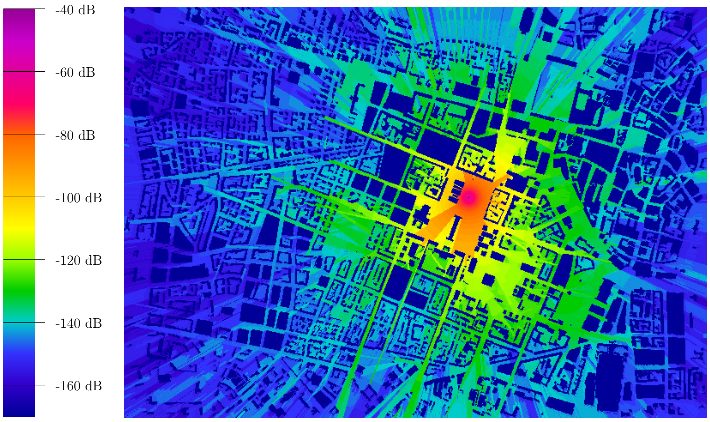
Finally, a random field $X(\boldsymbol{x})$ is called *Gaussian* if all finite dimensional marginals are jointly Gaussian distributed.

An extension of model (1) to spatial parameters would be as follows. The shadowing path gain from a transmitter located at position $\boldsymbol{b} \in \mathbb{R}^2$ to a receiver at position $\boldsymbol{x} \in \mathbb{R}^2$ is modeled by a random field $\{G_{\boldsymbol{b}}(\boldsymbol{x})\}$ with

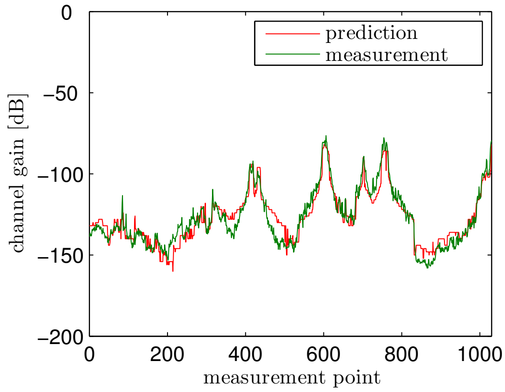
$$G_{\boldsymbol{b}}(\boldsymbol{x}) = g(\|\boldsymbol{x} - \boldsymbol{b}\|)\, G_{\mathrm{shad}}(\boldsymbol{x}).$$

Function $g(\cdot)$ represents the distance dependend path gain and further constants like antenna gains and frequency dependent attenuation. $\{G_{\mathrm{shad}}(\boldsymbol{x})\}$ is a random field defined by

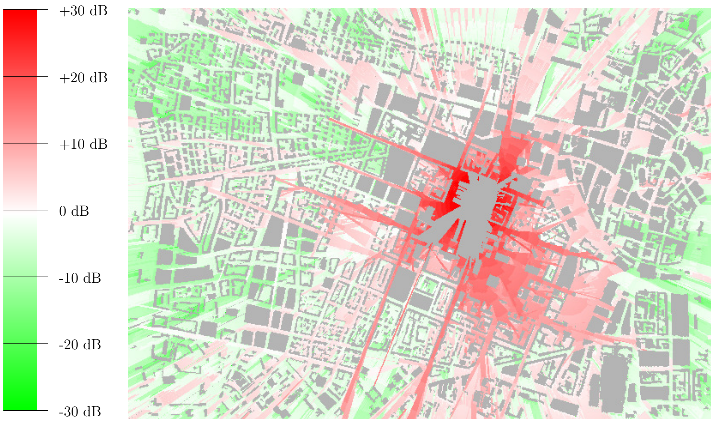
$$G_{\mathrm{shad}}(\boldsymbol{x}) = 10^{G(\boldsymbol{x})/10}.$$



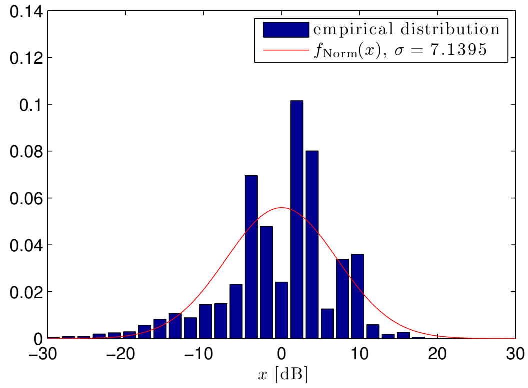
(a) CORLA Prediction.



(b) Comparison of measurement and prediction.



(c) Slow fading contribution in Figure 1(a).



(d) Empirical distribution of the shadowing and (Log-) Normal approximation.

Fig. 1. Analysis of path gain predictions in Munich, resolution 5m, frequency 947 MHz, transmitter position and building data from COST 231.

$\{G(\boldsymbol{x})\}$ is assumed as a zero mean, weak sense stationary and isotropic Gaussian random field with $G(\boldsymbol{x}) \sim \mathrm{N}(0, \sigma^2)$ for all $\boldsymbol{x}$. Further, we assume an exponential correlation function for $\{G(\boldsymbol{x})\}$ of the form

$$R_G(\tau) = \vartheta_1^{\tau^{\vartheta_2}} \tag{4}$$

with $\tau = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$ and parameters $\vartheta_1 \in (0, 1)$, $\vartheta_2 > 0$. An exponential correlation function is also considered in [1] for one-dimensional correlated shadow fading.

Under these model assumptions it can be shown that the random field $\{G_{\mathrm{shad}}(\boldsymbol{x})\}$ is isotropic, and all one-dimensional marginals are log-normal, i.e., $G_{\mathrm{shad}}(\boldsymbol{x}) \sim \mathrm{LogN}(0, \sigma^2)$ for all $\boldsymbol{x}$. Furthermore, the correlation function of $\{G_{\mathrm{shad}}(\boldsymbol{x})\}$ is given by

$$R_{G_{\mathrm{shad}}}(\tau) = \frac{\exp\left((\sigma \ln(10)/10)^2 \vartheta_1^{\tau^{\vartheta_2}}\right) - 1}{\exp\left((\sigma \ln(10)/10)^2\right) - 1}.$$

It remains to estimate the parameters ϑ_1 and ϑ_2. Resembling the empirical correlation function of a stochastic process with one-dimensional index space we define the estimator

$$\hat{R}_{G,r}(\tau) = \frac{1}{|\mathcal{M}_r(\tau)|} \sum_{(\boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathcal{M}_r(\tau)} \frac{G(\boldsymbol{x}_1) G(\boldsymbol{x}_2)}{\sqrt{\mathrm{Var}(G(\boldsymbol{x}_1)) \mathrm{Var}(G(\boldsymbol{x}_2))}},$$

where $\mathcal{M}_r(\tau)$ is a finite set of sample pairs

$$\mathcal{M}_r(\tau) = \{(\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}) \mid \|\boldsymbol{x}_{i1}\|, \|\boldsymbol{x}_{i2}\| \leq r, \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| = \tau, \; i = 1, \ldots, M(\tau)\},$$

and $G(\boldsymbol{x}_i)$ are observed values of the field G at position $\boldsymbol{x}_i$.

Figure 2 depicts $\hat{R}_{G(\boldsymbol{x}),\infty}(\tau)$ for the CORLA path gain predictions in Figure 1(a). The parameters ϑ_1 and ϑ_2 are determined as the solution of the optimization problem

$$\min_{\vartheta_1 \in (0,1),\, \vartheta_2 > 0} \sum_{\tau} \left(\hat{R}_{G(\boldsymbol{x}),\infty}(\tau) - \vartheta_1^{\tau^{\vartheta_2}}\right)^2.$$

for a finite set of points τ.

Solving this problem for the exponential correlation function yields

$$\hat{\vartheta}_1 = 0.9966 \quad \text{and} \quad \hat{\vartheta}_2 = 0.9682.$$

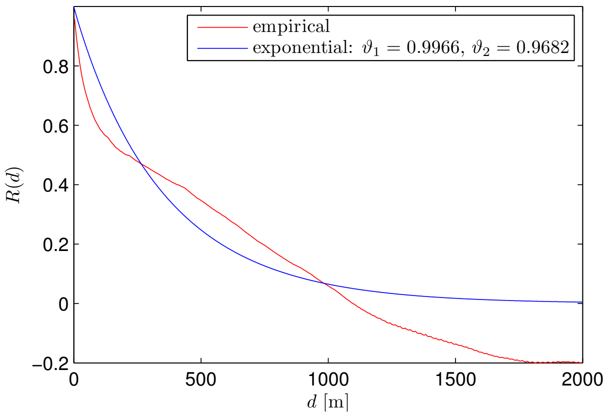


Fig. 2. Empirical correlation function and its approximation for the situation depicted in Figure 1(a).

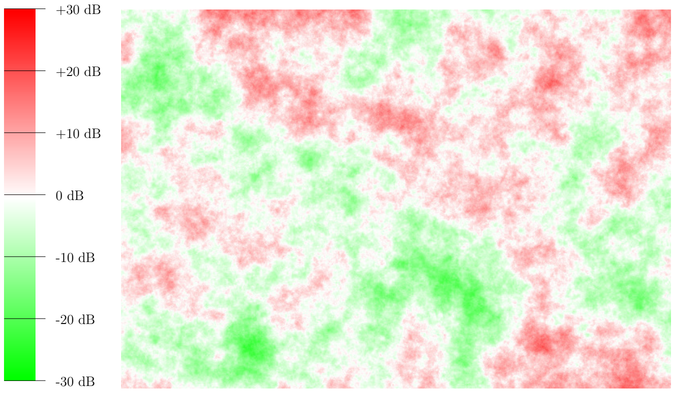


Fig. 3. Gaussian random field with exponential correlation function (4) and parameters $\sigma = 7.14$, $\vartheta_1 = 0.9966$ and $\vartheta_2 = 0.9682$.

yielding an acceptable approximation depicted in Figure 2. Deviations between the approximation and the empirical function for large distances may be explained by the fact that only a few points with a distance of more than 1000 m occur in the considered scenario. Furthermore, the correlation function is consistent with results obtained in the one-dimensional case, see [1].

Efficient algorithms for creating samples of Gaussian random fields are available, see [9]. Figure 3 depicts a realization of $\{G(\boldsymbol{x})\}$ for the above estimated parameter values using the tools from [9]. Structural similarities between the prediction in Figure 1(c) and the Gaussian random field in Figure 3 are clearly observable. Connected regions of positive or negative fading for example tend to be of similar size in both pictures. Figure 4 depicts the distance dependent path loss together with correlated shadow fading, i.e., $g(\|\boldsymbol{x}-\boldsymbol{b}\|)\,G_{\text{shad}}(\boldsymbol{x})$ on a dB scale. This figure represents a sample output of the considered channel model excluding fast fading.

Observe that the data is generated on the basis of a generic stochastic model and does not take account of concrete building data. It hence must not be expected that a typical urban

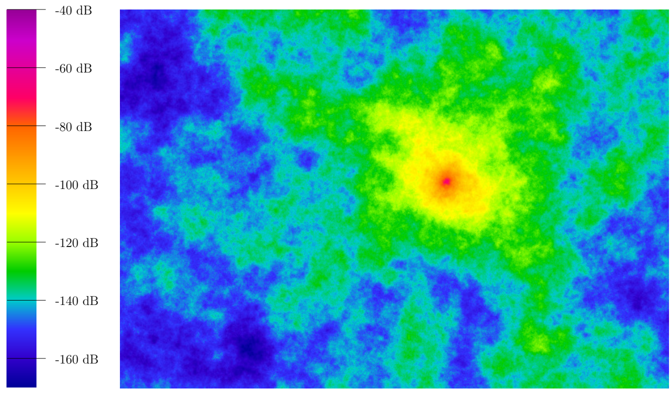


Fig. 4. Path gain including the distance dependent part and the shadow fading generated by a Gaussian random field with exponential correlation function (4) and parameters $\alpha = 3.57$, $\sigma = 7.14$, $\vartheta_1 = 0.9966$ and $\vartheta_2 = 0.9682$.

path gain pattern of the type in Figure 1(a) is generated.

V. Correlations between Base Stations

So far, we have considered spatial correlations in the path gain at different receiver positions from the same base station. However, for handoff analysis purposes correlations between the channel gains from different base stations to a single receiver location are of particular interest. Models for this type of correlations are an active area of research, e.g., within the IST Information Society Technologies-WINNER Wireless World Initiative New Radio II project [10]. We present a first modeling approach, which will be enhanced by spatial correlation components in the future.

As in the work [11], we consider only the distance dependent path gain and shadow fading. Assume K base stations at distances $d_1,\dots,d_K$ from a receiver. The path gains are described by the random vector

$$\boldsymbol{G}_{d_1,\dots,d_K} = \big(g(d_1)\,G_{\text{shad},1}\,G_{\text{shad,cor}},\dots,g(d_K)\,G_{\text{shad},K}\,G_{\text{shad,cor}}\big)'$$

with $g(\cdot)$ defined as above and $K+1$ stochastically independent random variables $G_{\text{shad,k}}$, $k=1,\dots,K$. We further assume that

$$G_{\text{shad,cor}} \sim \text{LogN}(0,\sigma_{\text{c}}^2)$$
$$G_{\text{shad},k} \sim \text{LogN}(0,\sigma_{\text{i}}^2)$$

for $k=1,\dots,K$. The random variable $G_{\text{shad,cor}}$ creates the correlation between different stations, whereas $G_{\text{shad},k}$, $k \in \{1,\dots,K\}$ models stochastically independent contributions to the path gain. To be consistent with the model introduced in Section II we require that

$$G_{\text{shad},k}\,G_{\text{shad,cor}} \sim \text{LogN}(0,\sigma^2)$$

for $k=1,\dots,K$, that is,

$$\sigma^2 = \sigma_{\text{i}}^2 + \sigma_{\text{c}}^2.$$

Thus, there is an $\eta \in [0,1]$ with $\sigma_{\text{i}}^2 = (1-\eta)\sigma^2$ and $\sigma_{\text{c}}^2 = \eta\,\sigma^2$. In [11] the value $\eta = 1/2$ was suggested.

VI. Conclusions and Outlook

Gaussian random fields are a valuable model for the simulation of spatially correlated log-normal fading. Even for the simple case of isotropic correlation functions, good approximatons have been obtained in this work. Comparing random fading maps with predicted ones, structural similarities have been detected. Future work will be devoted to including non isotropic correlation functions. Furthermore correlations between base stations will be extended by a spatial component in subsequent work. Finally, we will integrate the present model into a full simulation environment which supports investigating channel impairments and upper layer aspects for moving stations in complicated correlated fading scenarios.

References

[1] M. Gudmundson, "Correlation model for shadow fading in mobile radio systems," *IEE Electronics Letters*, vol. 23, no. 27, pp. 2145–2146, November 1991.

[2] K. Kumaran, S. E. Golowich, and S. Borst, "Correlated shadow-fading in wireless networks and its effect on call dropping," *Wireless Networks*, vol. 8, no. 1, pp. 61–71, January 2002.

[3] A. Algans, K. I. Pedersan, and P. E. Mogensen, "Experimental analysis of the joint statistical properties of azimuth spread, delay spread, and shadow fading," *IEEE J. Select. Areas Commun.*, vol. 20, no. 3, pp. 523–531, April 2002.

[4] A. Mawira, "Models for the spatial correlation functions of the (log)-normal component of the variability of VHF/UHF field strength in urban enviroment." in *Proc. Personal, Indoor and Mobile Radio Communications (PIMRC'92)*, 1992, pp. 436–440.

[5] B. Sorensen, "Correlation model for slow fading in a small urban macro cell," in *Proc. Personal, Indoor and Mobile Radio Communications (PIMRC'98)*, 1998, pp. 1161–1165.

[6] M. Schmeink, "Optimierungsalgorithmen zur automatisierten Funknetzplanung." Ph.D. dissertation, RWTH Aachen University, 2005.

[7] E. Damosso, Ed., *COST Action 231: Digital mobile radio towards future generation systems, Final Report*. Luxembourg: Office for Official Publications of the European Communities, 1999.

[8] E. VanMarcke, *Random Fields: Analysis and Synthesis*, 1998th ed. The MIT Press Cambridge, Massachusetts London, England, 1998.

[9] B. Kozintsev, "Computations with Gaussian random fields," Ph.D. dissertation, Faculty of the Graduate School of the University of Maryland, 1999.

[10] IST, "WINNER - Wireless world initiative new radio." [Online]. Available: http://www.ist-winner.org

[11] A. Viterbi, A. Viterbi, and E. Zehavi, "Other-cell interference in cellular power-controlled CDMA," *IEEE Trans. Commun.*, vol. 42, no. 2,3,4, pp. 1501–1504, February/March/April 1994.