Generalised water-filling: costly power optimally allocated to sub-carriers under a general concave performance function

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***Abstract*—A data terminal seeks the optimal power allocation for M sub-carriers under a *general* concave performance function ("capacity"), and a per-Watt price. The value to the terminal of one transferred information bit, as well as the noise-normalised channel gains for each sub-carrier are the critical parameters. We provide the structure of the general solution, and give closed-form expressions for the logarithmic form. The analysis reveals that if a sub-channel gain is less than or equal to the suitably normalised power price said sub-channel is not usable. Our results are directly applicable for power allocation in a time-division OFDM system. They can also be useful in an OFDMA system, if complemented with some sensible sub-channel allocation scheme, as we propose separately.**

I. Introduction

In an orthogonal frequency-division multiplexing (OFDM) system, a data-transferring terminal seeks the optimal amount of power for each of M sub-carriers, under a total power constraint, and a per-Watt price. The water-filling solution to this type of problems with costless power is well-known (for example, see Section 5.3.3 in [1]). However, when power is costly, the problem seems much less explored. Theorem 1 from [2] is the most relevant contribution of which we are aware, but it only addresses a specific logarithmic performance function, and it is stated without proof.

Below we formally solve the problem for a *general* concave performance function (see figure 1), and give closed-form solutions for the logarithmic special case. Our motivation is to apply our results within a decentralised, market-oriented resource-allocation scheme for multi-user OFDM [3]. However, we believe that the analysis has independent value, and in particular our results are directly applicable for price-based power allocation in a time-division OFDM system.

II. Definitions/Notation

Let M be the number of sub-carriers; $h_m > 0$ be the transmitter-receiver channel gain over sub-carrier m divided by the average noise level, c_0 the price per unit of power, and b_0 the value of one transferred information bit. c_0 and b_0 can be interpreted as "true" economic values measured in a pertinent monetary unit.

For convenience, and without loss of generality, we assume that the channel gains are labelled so that $h_1 \geq \cdots \geq h_M > 0$.

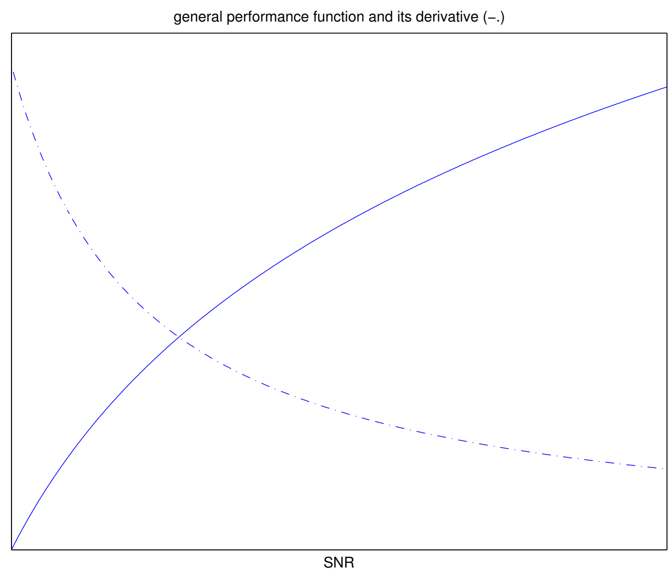

Figure 1. A performance ("capacity") function and its derivative (dash)

Let $p_m \geq 0$ be the amount of power allocated to sub-carrier m, and $x_m := h_m p_m$ (the signal-to-noise ratio (SNR) at the receiver)

The terminal wishes to solve the problem :

$$\begin{aligned} &\max_{x_1,\cdots,x_M} b_0 \sum_{m=1}^{M} f_0(x_m) - c_0 \sum_{m=1}^{M} \frac{x_m}{h_m} \\ &\text{s.t.} \sum_{m=1}^{M} \frac{x_m}{h_m} \leq P \\ &x_m \geq 0 \end{aligned} \tag{1}$$

Definition 1. $f_0 : \Re_+ \to \Re_+$ is strictly increasing, concave, and satisfy (i) $f_0(0) = 0$, (ii) $f_0'(0) < \infty$, and (iii) $\lim_{t\to\infty} f_0'(t) = 0$ (see figure 1).

Remark 2. Since f_0 is concave, f_0'' is negative, and therefore f_0' is strictly decreasing. In particular, $f_0'(0) > f_0'(t) \quad \forall t > 0$.

Definition 3. $f_S(t) := \ln(1+t)$

For reasons that may later become clear, the objective function can be re-written as

$$b_0 f_0'(0)\left(\sum_{m=1}^{M}\frac{f(x_m)}{f_0'(0)} - \frac{c_0}{b_0 f_0'(0)}\sum_{m=1}^{M}\frac{x_m}{h_m}\right) \tag{2}$$

Since the $b_0 f_0'(0)$ factor will not change the optimiser, with $c := c_0/(b_0 f_0'(0))$ and $f(t) := f_0(t)/f_0'(0)$ the terminal's problem can be re-stated as:

$$\begin{aligned} \max_{x_1,\cdots,x_M} & \sum_{m=1}^{M} f(x_m) - c\sum_{m=1}^{M}\frac{x_m}{h_m} \\ \text{s.t.} & \sum_{m=1}^{M}\frac{x_m}{h_m} \le P \\ & x_m \ge 0 \end{aligned} \tag{3}$$

Remark 4. f retains the properties of f_0 (Definition 1, Remark 2) with the additional specification that $f'(0) = 1$, by definition of f.

III. KKT CONDITIONS AND SOLUTIONS

A. KKT FONOC

Fact 5. *If $(x_1^*, \cdots, x_M^*)$ is a (local) optimiser corresponding to Problem (1), then there are non-negative real numbers $\lambda, \mu_1, \cdots .\mu_M$ such that*

$$h_m f'(x_m^*) = c + \lambda - \mu_m \quad \forall m \tag{4}$$

$$\lambda\left(\sum_{m=1}^{M}\frac{x_m^*}{h_m} - P\right) = 0 \tag{5}$$

$$\mu_m \frac{x_m^*}{h_m} = 0 \quad \forall m \tag{6}$$

Remark 6. Fact 5 follows directly from the well-known Karush-Kuhn-Tucker (KKT) first-order necessary optimising conditions (FONOC) [4], [5].

Remark 7. μ_m arises from the non-negativity constraint $x_m \ge 0$. The denominator in (6) is for algebraic convenience.

Lemma 8. *Conditions (4) and (6) can be replaced by the equivalent condition:*

$$h_m f'(x_m^*) \le c + \lambda \quad \forall m \quad \textit{with equality if } x_m^* > 0 \tag{7}$$

Proof: (i) Since $\mu_m \ge 0$, $h_m f'(x_m^*) + \mu_m = c + \lambda \implies h_m f'(x_m^*) \le c + \lambda$;

(ii) If $x_m^* > 0$, (6) requires $\mu_m = 0$ which, with (4), further implies that $h_m f'(x_m^*) = c + \lambda$ ■

B. Solutions to the FONOC

Definition 9. With $S := \{m : h_m \le c\}$, let

$$m_0 := \begin{cases} M+1 & \text{if } S = \emptyset \\ \min(S) & \text{otherwise} \end{cases}$$

Lemma 10. *If $(x_1^*, \cdots, x_M^*)$ is a (local) optimiser of Problem (1) then $(m_0 \le m \le M) \implies x_m^* = 0$.*

Proof: By convention, $h_m \ge h_{m+1}\,\forall m$; thus, $h_{m_0} \le c \implies h_m \le c\,\forall m > m_0$. Furthermore, since $\lambda \ge 0$, $h_m \le c \implies h_m < c + \lambda$.

By Remark 4, f' is decreasing, with $f'(0) = 1$. Thus $h_m \le c + \lambda \implies h_m f'(t) < c + \lambda\,\forall t > 0$, and this implies that there exists *NO* $t > 0$ that can satisfy the necessary condition $h_m f'(t) = c + \lambda$ (conditions (4) and (6) together). ■

Remark 11. Lemma 10 indicates that a general solution to (4) — (6) allocates power to at most the first $m_0 - 1 \le M$ sub-carriers. The quantity c, the normalised power cost, defines a threshold for channel usability; if $h_m \le c$ then the sub-channel is useless.

Below we discuss separately two cases: (i) $\lambda = 0$ and (ii) $\lambda > 0$ (respectively, "plentiful" and "scarce" power).

1) *Plentiful power ($\lambda = 0$):*

Lemma 12. *If a non-negative vector $(x_1^*, \cdots, x_M^*)$ is such that (i) $x_m^* = 0 \quad \forall m \in \{m_0, \ldots, M\}$, (ii)*

$$h_m f'(x_m^*) = c \quad \forall m \in \{1, \ldots, m_0 - 1\} \tag{8}$$

and (iii) $\sum_{m=1}^{M}(x_m^*/h_m) \le P$ then $(x_1^*, \cdots, x_M^*)$ satisfies conditions (4), (5) and (6).

Proof: It can be verified by direct substitution that, under the hypothesis, the choice $\lambda = \mu_1 = \cdots = \mu_{m_0-1} = 0$, and $\mu_m = c - h_m \quad \forall m \in \{m_0, \ldots, M\}$, satisfies conditions (4), (5) and (6). ■

Remark 13. For the specific case in which $f = f_S$, the solution to (8) is given by

$$p_m^* + \frac{1}{h_m} = \frac{1}{c} \quad \forall m \in \{1, \ldots, m_0 - 1\} \tag{9}$$

("water filling" for the first $m_0 - 1$ sub-carriers).

2) *Scarce power ($\lambda > 0$):*

Lemma 14. *If there exist $\lambda > 0$ and non-negative $(x_1^*, \cdots, x_M^*)$ such that (i)*

$$x_m^* = \begin{cases} 0 & \textit{if } h_m \le c + \lambda \\ h_m f'(x_m^*) = c + \lambda & \textit{otherwise} \end{cases} \tag{10}$$

and (ii) $\sum_{m=1}^{M}(x_m^*/h_m) = P$, then $(x_1^*, \cdots, x_M^*)$ satisfies conditions (4), (5) and (6).

Proof: Under the hypothesis, the choice

$$\mu_m = \begin{cases} c + \lambda - h_m & \text{if } h_m \le c + \lambda \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

satisfies conditions (4), (5) and (6). ■

Remark 15. By Definition 1, f' is monotonic and hence invertible (see figure 1). Thus, $x_m(\lambda) := f'^{-1}((c+\lambda)/h_m)$ yields x_m as a function of λ. If f'^{-1} can be expressed in closed form, replacing $x_m(\lambda)$ into the power constraint yields a single-variable equation which can be solved for λ. This is done below for $f = f_S$. Notice also that, since $\lambda > 0$, $h_m \le c \implies h_m < c + \lambda$, thus the sub-channels for which $h_m \le c$ remain unusable, and need not be considered. At most $m_0 - 1$ sub-channels are used.

Remark 16. For $f = f_S$, equation (10) yields $x_m/h_m = \max(0, 1/(c+\lambda) - 1/h_m)$. For $1 \le m \le m_0 - 1$, $h_m > c$, but depending on the value of λ, h_m could be less than $c + \lambda$

for some m. Assuming first that for the resulting value of λ, $h_{m_0-1} \geq c+\lambda$ the power constraint leads to

$$\frac{m_0-1}{c+\lambda} - \sum_{m=1}^{m_0-1} \frac{1}{h_m} = P \tag{12}$$

or

$$c+\lambda = \frac{m_0-1}{P+\sum_{m=1}^{m_0-1} \frac{1}{h_m}} \tag{13}$$

If h_{m_0-1} is greater than or equal to the right side of (13), and this is greater than c, then the corresponding value of λ is accepted. Otherwise, x_{m_0-1} is set to zero (sub-channel m_0-1 is abandoned), and the previous step is repeated considering the first m_0-2 channels only: $c+\lambda = (m_0-2)/(P+\sum_{m=1}^{m_0-2} 1/h_m)$. And so forth. The procedure ends with an appropriate λ^*, and only the first $M_1 \leq m_0-1$ receiving positive power.

Thus, (10) ultimately yields to "water filling" with the adjusted power price $c+\lambda^*$:

$$p_m^* + \frac{1}{h_m} = \frac{1}{c+\lambda^*} \quad \forall m \in \{1,\ldots,M_1\} \tag{14}$$

IV. Discussion

The sub-channel power allocation problem and its "water-filling" solution are well-known, when the performance ("capacity") function is logarithmic and power is constrained but costless. We have generalised this problem by considering a general concave performance function and a per-Watt price. Our analysis is directly applicable as a price-driven power-allocation solution, for example, in a time-division OFDM scenario. It can also be useful in an OFDMA scenario, if combined with an appropriate sub-channel allocation scheme (as we propose separately [3]).

The solution retains the general water-filling structure, but the costly power does change matters significantly. In particular, suitably normalised, the power price translates to a channel gain threshold for channel usability: if a noise-normalised sub-channel gain fails to exceed the normalised power price, the sub-channel is not usable. Conversely, the highest sub-channel gain defines a maximum (normalised) power price, beyond which no sub-channel is used. The intuition is clear: when a sub-channel gain is sufficiently low (or the normalised power price is sufficiently high), the marginal benefit of buying a unit of power is less than its cost. It is possible that all sub-channels be discarded for that reason.

When the power is plentiful, each usable sub-channel can be allocated its individual maximiser ("greedy" solution); that is, the choice that maximises benefit minus cost on a channel-by-channel basis. The "greedy" optimiser is defined by $h_m f'(h_m p_m) = c$, which itself has a "water-filling" interpretation in logarithmic ("dB") scale: $\log(h_m) + \log(f'(h_m p_m)) = \log(c)$. Of course, the power levels resulting from individual sub-channel optimisation may exceed the total power constraint ("scarce power"), in which case , λ, the corresponding Lagrange multiplier, becomes non-zero. Its net effect is that the system behaves as if the power price was $c+\lambda$ ("effective" price): $h_m f'(h_m p_m) = c+\lambda$. At the optimum, $c+\lambda$ takes the value necessary to make total power consumption ("demand") equal to the total available "supply" (i.e., the "market clearance" price). Thus, some sub-channels that would be usable at the true price ($h_m > c$), may become unusable under the "effective price": $h_m \leq c+\lambda$.

In the development, closed-form solutions are given for the special case $f(t) = \ln(1+t)$ leading to water-filling over the usable sub-channels: $p_m + 1/h_m = 1/c$ when power is plentiful, and $p_m + 1/h_m = 1/(c+\lambda)$ otherwise.

We have not yet checked the second-order sufficient conditions, but it is intuitively clear that at least the plentiful-power solution is globally optimal: each *usable* sub-channel is allocated its individual optimiser; the terminal cannot possibly do better. We have not discussed the constraint qualification condition, but some reflection indicates that such condition is always satisfied. These issues will be addressed in greater detail in future reports of this work.

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