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Strategies for Source/Sink Pairs in Amplify-and-Forward Gaussian Relay Networks

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***Abstract*—This paper presents a framework for studying amplify-and-forward Gaussian relay networks and optimizing the strategy of source/sink pairs. A typical communication scenario is that many pairs of users try to exchange information over the same network. Changing the strategy of the relays, in such a network, for each communication might be an overwhelming task. We consider the case where the behavior of the relays is fixed. By solely optimizing the strategy of the source and the sink, the goal is to maximize the mutual information between the transmitted signal and the received signal. We show a parallel between the multiple-input multiple-output (MIMO) single user channel and amplify-and-forward Gaussian relay networks and derive solutions for three different types of power constraints on the network, namely 1) source power constraint, 2) global network power constraint and 3) individual relay power constraints. Finally we provide numerical results for an example network.**

I. Introduction

Relay networks model communication problems where one or more sources transmit information to one or more sinks through relays which aim at optimizing the whole communication process. The simple relay network with one source, one sink and one relay was introduced in [1]. Fundamental results on this single relay channel are presented in [2]. Another example of a basic network is the Gaussian parallel network, introduced in [3], where one source communicates to two relays over two channels that do not interfere. The relays then forward information to the sink.

In [4] and [5] the capacity of a more general network is studied. In this network a source transmits a signal to a sink through one direct path and M relays. The direct path signal and the M signals forwarded by the relays interfere before to reach the sink. In [6], the authors compare decode-and-forward and amplify-and-forward strategies for this network and derive optimal regions for each strategy in the case of a one relay network. In [7], the authors further study capacity bounds of this network while letting the number of nodes going to infinity.

There exists other capacity results, for general network topologies, which can be found for example in [8]–[10]. In [8], the authors derive capacity regions while using multi-hopping over a network of arbitrary size and the work in [9], [10] is concerned with finding cooperative coding strategies to develop achievable rate regions.

This paper presents a framework for amplify-and-forward Gaussian relay networks and our approach is different from these fundamental results in several aspects. The main difference consists in the way we approach the relaying strategy. Generally one wants to optimize the strategy of the relays to maximize the mutual information between the signal of the transmitter and the receiver. We use our framework to consider networks where the relaying strategy is fixed or random but known in advance. This is motivated by the fact that relay networks will be used by many pairs of source and sinks, recalculating and changing strategies for each transmission is a daunting task. Therefore we assume that the relays use a generic strategy for any communication, which is not optimized for a precise source/sink pair. What will be optimized is the strategy of the source and the sink. Note nevertheless that it is possible to optimize the strategy of the relays using our model. This is however out of the scope of this present paper.

Another notable specificity of this work is the algebraic model underlying the network topology. This model makes apparent a strong connection between the multiple-input multiple output (MIMO) single user channel and amplify-and-forward relay networks. This link enables the use of powerful methods developed in MIMO theory. The fundamental model of the single user MIMO channel (which can be found in [11], [12]) is recalled for later comparison and is described as

$$\hat{\mathbf{x}} = \mathbf{W}^{\mathrm{H}}\mathbf{H}\mathbf{P}\mathbf{x} + \mathbf{W}^{\mathrm{H}}\boldsymbol{\eta} \tag{1}$$

where the transmitter has n_T antennas, the receiver has n_R antennas, $\mathbf{x} \in \mathbb{C}^L$ is the transmitted signal, $\hat{\mathbf{x}} \in \mathbb{C}^L$ is the received signal, $\mathbf{P} \in \mathbb{C}^{n_T \times L}$ is a precoding matrix at the transmitter, $\mathbf{H} \in \mathbb{C}^{n_R \times n_T}$ is the channel matrix, $\mathbf{W} \in \mathbb{C}^{n_R \times L}$ is a filter at the receiver and $\boldsymbol{\eta} \in \mathbb{C}^{n_R}$ is noise vector with entries identically-distributed taken from the Gaussian distribution. The power constraint of this system is expressed as $\mathrm{Tr}(\mathbf{P}\mathbf{P}^{\mathrm{H}}) \leq P_T$, where P_T is the available power at the transmitter.

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A last interesting connection with relay networks is network coding [13], [14]. In a coded network, relays can not only forward incoming packets but also combine them linearly before to forward them. Although network coding considers symbol transmission over a finite field and assumes edges with perfect reliability, the algebraic matricial structure of the network coding problem can be transposed to relay networks problems.

In Section II, we recall the algebraic model of [14]. In Section III we present our model of the network. In Section IV, we show the connexion between the MIMO single user channel and amplify-and-forward relay networks. In Section V we derive the optimal filter for single-source single-sink networks. We consider three different power constraints (namely, power constraint only on the source, global constraint for the whole network and individual power constraints for each node). We derive the optimal precoder at the source for the first two constraints and present a greedy algorithm for the last one. In Section VI, we present numerical results for an example network. Finally Section VII wrap-ups this work.

II. Algebraic Model for Ideal Networks

In [14], the author proposed an algebraic model for networks based on directed graphs as a framework for network coding over finite alphabets. We want to extend this model to Gaussian relay networks (although our model is continuous, we can reuse the way matrices model a network in [14]). We first present the basis from the model of [14]. An example of a coded network is illustrated in Figure 1.

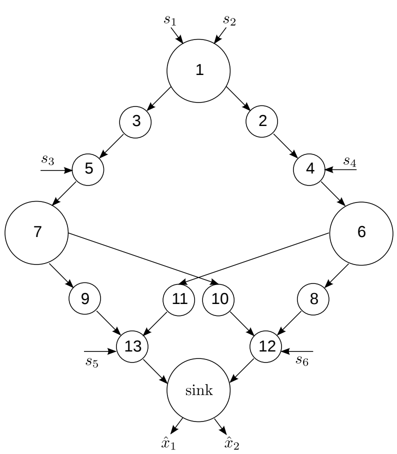

Fig. 1: Ideal network.

In this example, the symbol s_i represent a signal component transmitted by the node it is attached to. In other words, node 1 transmits two signal components, and nodes $4, 5, 12$ and 13 one signal component each. In this network all transmitted nodes communicate with the sink. Therefore a signal $\mathbf{s} = (s_1, \ldots, s_6)$ is transmitted to the sink. The symbol $\hat{x}_i$ represents a signal component received by the sink, in this example, the sink receives a signal $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$.

In this framework, edges are perfectly reliable. An important point is that two edges arriving at the same node do not interfere. Furthermore, intermediate nodes can linearly combine incoming data before to forward them. Such network can be represented as a linear system,

$$\hat{\mathbf{x}} = \mathbf{G}\mathbf{s}, \tag{2}$$

where $\mathbf{s}$ is the input vector of size m, $\hat{\mathbf{x}}$ is the output vector of size n and $\mathbf{G}$ is a $n \times m$ matrix modeling the network. In [14] it is shown that $\mathbf{G}$ can be represented with three matrices $\mathbf{A}$, $\mathbf{F}$ and $\mathbf{B}$. The matrix $\mathbf{A}$ of size $e \times m$ (e is the number of edges in the network) represents the linear coefficients $\tilde{a}_{ij}$ chosen by the source node to send information through the network. The matrix $\mathbf{A}$ has the form

$$a_{ij} = \begin{cases} \tilde{a}_{ij} & \text{if } x_j \text{ can be sent directly through edge } i \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The matrix $\mathbf{F}$ of size $e \times e$ represents the coefficients $\tilde{f}_{ij}$ chosen by intermediate nodes while forwarding messages. The matrix $\mathbf{F}$ has the form

$$f_{ij} = \begin{cases} \tilde{f}_{ij} & \text{if a bit can flow directly from edge } j \text{ to } i \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Finally the matrix $\mathbf{B}$ of size $e \times n$ represents the coefficients $\tilde{b}_{ij}$ chosen by the sink to filter information out of the network.

$$b_{ij} = \begin{cases} \tilde{b}_{ij} & \text{if } \hat{x}_j \text{ can be received directly from edge } i \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Note that $\mathbf{F}$ represents the amplification of the signal after one hop. Therefore $\mathbf{F}^k$ represents the amplification of the signal after k hops. Interestingly $\mathbf{F}$ is strictly lower diagonal and is therefore a nilpotent matrix, so there exists a power q such that $\mathbf{F}^q = \mathbf{0}$. It is possible to represent the amplification of the signal after between zero and $q-1$ hops, with only one lower diagonal matrix $\mathbf{M}$ of size $e \times e$,

$$\mathbf{M} = \mathbf{I}_e + \mathbf{F} + \mathbf{F}^2 + \cdots + \mathbf{F}^{q-1} = (\mathbf{I}_e - \mathbf{F})^{-1}.$$

It is proved in [14] that

$$\hat{\mathbf{x}} = \mathbf{B}^{\mathrm{H}}\mathbf{M}\mathbf{A}\mathbf{s} = \mathbf{B}^{\mathrm{H}}(\mathbf{I}_e - \mathbf{F})^{-1}\mathbf{A}\mathbf{s}. \tag{6}$$

Note that the only similarity between our work and the work of [14] is the way a network is modeled using matrices. We now extend this model to Gaussian relay networks.

III. System Model and Optimization Problem for Gaussian Relay Networks

We consider networks that can be represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. For simplicity, we only consider the case where the network has one source, one sink and arbitrarily many relays. Note that we present in the Appendix an extension to this model which represents networks with several sources and several sinks. In this framework the relays can 1) linearly combine incoming signals and 2) amplify-and-forward signals to the next relay or sink. We do not consider decode-and-forward strategies. All links in the network are noisy, with the noise being taken from the Gaussian distribution. We call h_i the channel gain of edge i.

The source wants to transmit a vector $\mathbf{x} \in \mathbb{C}^n$ to the sink which receives a vector $\hat{\mathbf{x}} \in \mathbb{C}^n$. Typically we want to maximize the mutual information between the transmitted signal $\mathbf{x}$ and the received signal $\hat{\mathbf{x}}$. The min-cut of the network is assumed to be at least n. It means that the network has enough degrees of freedom to transport the whole signal. We call $\boldsymbol{\eta} \in \mathbb{C}^N$ the noise vector applied to the edges of the network. We assume that $\mathbf{x}$ is zero-mean, normalized and uncorrelated such that $\mathbf{x} \sim \mathcal{N}_{\mathbb{C}}(0, \mathbf{I}_n)$ ($\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{H}}] = \mathbf{I}_n$) and the noise is such that $\boldsymbol{\eta} \sim \mathcal{N}_{\mathbb{C}}(0, \mathbf{C}_{\boldsymbol{\eta}})$ where $C_{\boldsymbol{\eta}} \in \mathbb{C}^{N \times N}$ is the covariance matrix of the Gaussian noise ($\mathbb{E}[\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{H}}] = \mathbf{C}_{\boldsymbol{\eta}}$). An example of a network we would like to study is illustrated in Figure 2.

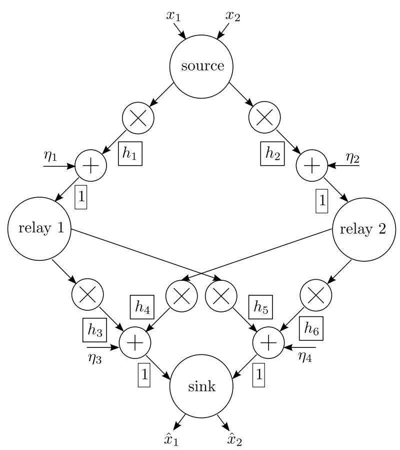

Fig. 2: Gaussian relay network.

In this example we have one source, one sink and two relays. Although we have only 4 real nodes, we represent also multiplicative nodes (nodes with a $\times$ symbol) and additive nodes (nodes with $+$ symbol). These nodes are virtual and only there for the sake of modeling channel gains and Gaussian noise. Typically a link between two real nodes is composed of three edges, one multiplicative node and one additive node. A multiplicative node multiplies an incoming signal with the channel gain of its link and an additive node adds a Gaussian noise component to an incoming signal. A simple link is illustrated in Figure 3.

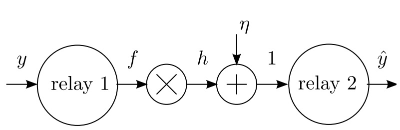

Fig. 3: A simple link inside a Gaussian relay network.

In this simple link we have $\hat{y} = hfy + \eta$, where y is a signal arriving at the first relay, f is the amplification applied to the signal by the first relay, h is the gain of the channel (applied by the multiplicative node), η is the Gaussian noise added to the signal and finally $\hat{y}$ is the signal received by the second relay.

To make our model clearer, we detail a bit more the example in Figure 2. We see that the source has two outgoing edges. It means that the signals sent on these two edges are totally orthogonal and do not interfere. In a wireless network it represents, e.g., the fact that a node transmits on two different frequency carriers. We can also see that several edges coming from different multiplicative nodes might arrive at the same additive node. It illustrates the fact that two signals interfere. As for our example, it would mean that the relays 1 and 2 both send two signals on the two same frequencies. The sink receives then two signals on two separate frequency carriers, each of them is a combination of a signal from relay 1 and 2. In general it is important to note that two signals on two edges arriving at the same node do not interfere and can be separated by the receiving node.

Note that the multiplicative and additive nodes could be put together as one single node, modeling the channel gains experienced by incoming signals as well as the Gaussian noise. In the rest of the present work we keep these two aspects separated for the sake of clarity.

The idea, in order to model mathematically such a network, is to apply the matricial framework of [14] to Gaussian relay networks. To embed a network like the one in Figure 2 in a framework that models networks like the one in Figure 1, we proceed to the following problem changes

1) The noise inputs are considered as part of the transmitted signal (although it will be filter later on).
2) The multiplicative nodes and additive nodes are considered as normal nodes in the sense of [14], i.e., they can amplify-and-forward incoming data.
3) The amplifying coefficient of multiplicative nodes is fixed and is equal to the channel gain of the link they are in.
4) The amplifying coefficient of additive nodes is fixed and is equal 1.

To model the first point we define $\mathbf{s} \in \mathbb{C}^{n+N}$ as the transmitted signal through the network (to relate to Section II $m = n + N$), where N is the number of noise inputs, i.e., $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N)$, where $\boldsymbol{\eta}$ is a vector composed of all noise inputs. Remember that $\mathbf{x} \in \mathbb{C}^n$, we have

$$\mathbf{s} = \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{pmatrix}. \tag{7}$$

The matrix $\mathbf{A}$ has now size $e \times (n + N)$. In our model $\mathbf{A}$ also models the coefficients to apply to the noise entering the network. All coefficients from a noise input to a network edge are set to one. We can always numerate the edges of the network such that the source can transmit the n components of $\mathbf{x}$ in the n first edges of the network. Therefore $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{\mathbf{x}} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\boldsymbol{\eta}} \end{bmatrix}, \tag{8}$$

where $\mathbf{A}_{\mathbf{x}}$ is a $n \times n$ matrix representing the coefficients applied from the source to the real signal $\mathbf{x}$ and $\mathbf{A}_{\boldsymbol{\eta}}$ is a $(e - n) \times N$ matrix composed of ones and zeros. In the matrix $\mathbf{F}$, the coefficients related to multiplicative nodes are set with the appropriate channel coefficients h_i. The coefficients related to additive nodes are set to one. The coefficients of $\mathbf{F}$ that remain unset represent the strategy chosen by the relays. We can always numerate the edges of the network such that the sink receives the n components of the incoming signal in the n last edges of the network. Therefore $\mathbf{B}$ has the form

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{B_x} \end{bmatrix}, \tag{9}$$

where $\mathbf{B_x}$ is a $n \times n$ matrix representing the filter applied to the received signal to get $\hat{x}$. Now all the parameters of the system are set, we can apply (6) to our Gaussian relay network ($\hat{\mathbf{x}} = \mathbf{B}^{\mathrm{H}}\mathbf{M}\mathbf{A}\mathbf{s}$) and write the optimization problem we want to solve. We maximize the mutual information between the source signal and the sink signal $I(\mathbf{x}, \hat{\mathbf{x}})$ with some power constraints on the network

$$\begin{aligned} \max_{\mathbf{A_x},\mathbf{B_x}}. \quad & I(\mathbf{x}, \hat{\mathbf{x}}) = I(\mathbf{x}, \mathbf{B}^{\mathrm{H}}\mathbf{M}\mathbf{A}\mathbf{s}) \\ \text{s.t.} \quad & \text{Power used in the network} \leq P_T. \end{aligned} \tag{10}$$

Note that we do not optimize our problem on $\mathbf{M}$ and consider it has fixed in this paper. Note also that we will study different power constraints as described later in Section VI.

IV. Parallels with MIMO model

In the problem (10), we optimize the matrices $\mathbf{A_x}$ and $\mathbf{B_x}$. However they are not apparent in (6). This can be changed expressing the matrix $\mathbf{M}$ as a block matrix as follow

$$\mathbf{M} = \begin{bmatrix} \mathbf{M_L} & \mathbf{M_R} \\ \mathbf{M_x} & \mathbf{M}_{\boldsymbol{\eta}} \end{bmatrix}, \tag{11}$$

where $\mathbf{M_x}$ is a $n \times n$ matrix corresponding to the part of the network that transports the input vector $\mathbf{x}$ form the source to the sink, $\mathbf{M}_{\boldsymbol{\eta}}$ is a $n \times (e-n)$ matrix that corresponds to the part of the network that transfers all the Gaussian noise to the sink. The matrices $\mathbf{M_L} \in \mathbb{C}^{(e-n)\times n}$ and $\mathbf{M_R} \in \mathbb{C}^{(e-n)\times(e-n)}$ represents respectively the part of the network that transports $\mathbf{x}$ and $\boldsymbol{\eta}$ to the relays. Now we can rewrite the received signal $\hat{\mathbf{x}}$ as

$$\hat{\mathbf{x}} = \mathbf{B_x}^{\mathrm{H}}(\mathbf{M_x}\mathbf{A_x}\mathbf{x} + \mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\boldsymbol{\eta}). \tag{12}$$

The similarities with a multiple antenna system are now apparent (comparing with (1)), $\mathbf{A_x}$ is in some sense a precoder at the source that enables to spread $\mathbf{x}$ over the different input edges and $\mathbf{B_x}$ is a filter that enables to separate the noise from the signal. The main differences are 1) that the noise is multiplied by $\mathbf{M}_{\boldsymbol{\eta}}$, in other words the noise is amplified inside the network, if the network is longer, the sink will receive more noise, 2) we can actually choose $\mathbf{M_x}$ and $\mathbf{M}_{\boldsymbol{\eta}}$ (in contrast to the channel matrix in multiple antenna systems) to improve communication. However, in this work, as explained in the introduction, we consider these matrices as fixed. An interesting parallel is also the fact that, the min-cut of the network is at least n, can interpreted in the MIMO sense as the fact that the receiver as at least as much antennas as the transmitter.

V. Source/Sink Strategy Optimization

As stated already we would like to maximize the mutual information between the transmitted signal and the received signal $I(\mathbf{x}, \hat{\mathbf{x}})$. We define the matrix $\mathbf{E}$ as the mean square error (MSE) matrix defined as $\mathbf{E} = \mathbb{E}[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^{\mathrm{H}}]$.

The matrix $\mathbf{E}$ depends on $\mathbf{A_x}$ and $\mathbf{B_x}$ and can be written as

$$\begin{aligned} \mathbf{E}(\mathbf{A_x}, \mathbf{B_x}) =& (\mathbf{B_x}^{\mathrm{H}}\mathbf{M_x}\mathbf{A_x} - \mathbf{I}_n)(\mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}}\mathbf{B_x} - \mathbf{I}_n) + \\ & \mathbf{B_x}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{B_x} \\ =& \mathbf{B_x}^{\mathrm{H}}(\mathbf{M_x}\mathbf{A_x}\mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}} + \\ & \mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}})\mathbf{B_x} + \\ & \mathbf{I}_n - \mathbf{B_x}^{\mathrm{H}}\mathbf{M_x}\mathbf{A_x} - \mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}}\mathbf{B_x}. \end{aligned} \tag{13}$$

Now our goal is to maximize $I(\mathbf{x}, \hat{\mathbf{x}})$ by choosing $\mathbf{A_x}$ and $\mathbf{B_x}$ appropriately.

A. Minimum MSE Filter Design

Methodologically we follow the work of [11] and [12]. We want to get the matrix $\mathbf{B_x}$ that minimizes the MSE between $\mathbf{x}$ and $\hat{\mathbf{x}}$, for that it suffices to minimize each $\mathbf{E}_{i,i}$ independently, since the choice of one b_i (i-th column of $\mathbf{B_x}$) only influence $\mathbf{E}_{i,i}$.

Theorem 1: The Wiener filter $\mathbf{B_x}^*$ that minimizes $\mathbf{E}_{i,i}$ for each $i = 1, \ldots, n$ is

$$\begin{aligned} \mathbf{B_x}^* =& (\mathbf{M_x}\mathbf{A_x}\mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}} + \mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}})^{-1}\mathbf{M_x}\mathbf{A_x} \\ =& (\mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}})^{-1}\mathbf{M_x}\mathbf{A_x} \\ & (\mathbf{I}_n + \mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}}(\mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}})^{-1}\mathbf{M_x}\mathbf{A_x})^{-1}. \end{aligned} \tag{14}$$

Proof: We simply set the gradient of $\mathbf{E}_{i,i}$ with respect to $\mathbf{b}_i$ to zero and find the optimal $\mathbf{b}_i$. The second form of $\mathbf{B_x}^*$ comes from the matrix inversion lemma, see [12] for calculation details. ■

By plugging (14) in (13) we obtain

$$\begin{aligned} \mathbf{E}(\mathbf{A_x}) =& \mathbf{I}_n - \mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}}(\mathbf{M_x}\mathbf{A_x}\mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}} + \\ & \mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}})^{-1}\mathbf{M_x}\mathbf{A_x} \\ =& (\mathbf{I}_n + \mathbf{A_x}^{\mathrm{H}}\mathbf{M_x}^{\mathrm{H}}(\mathbf{M}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}\mathbf{C}_{\boldsymbol{\eta}}\mathbf{A}_{\boldsymbol{\eta}}^{\mathrm{H}}\mathbf{M}_{\boldsymbol{\eta}}^{\mathrm{H}})^{-1} \\ & \mathbf{M_x}\mathbf{A_x})^{-1}, \end{aligned} \tag{15}$$

where $\mathbf{E}(\mathbf{A_x})$ is now the MSE matrix after Wiener filtering.

B. Precoder Design

From now on we still have to optimize the precoding matrix $\mathbf{A_x}$ to maximize the mutual information between $\mathbf{x}$ and $\hat{\mathbf{x}}$. We can express the mutual information as $I(\mathbf{x}, \hat{\mathbf{x}}) = -\log\det(\mathbf{E}(\mathbf{A_x}))$. To optimize $\mathbf{A_x}$ we need to consider the power constraints of the system, namely we will derive results for three different constraints. First we want to assume that the power at the source is limited but the power at the relays is unconstrained. It does not mean that the power at the relays is unlimited, namely the relays are power constrained but the source is not concerned with the power used at the relays. In other words, the relays use a certain fixed amplification and are designed to be capable of amplifying any input signal. This case illustrates a network where the source is a small, power limited device, transmitting to a large wireless relay, which forwards the signal to other relays or base stations until it can reach the sink. The available power at the relays is so much larger than the one at the source that we can simply drop the power constraints at the relays. The second scenario we will study, is the one of a cognitive network where all nodes are the same and share a global power constraint and, by allocating the power where its best used, try together to maximize $I(\mathbf{x},\hat{\mathbf{x}})$. Finally, in the last scenario that we consider, all nodes have an individual power constraint. In the last two cases, the source does not influence the strategy of the relays, it is responsible on not making them violate the constraints. In all cases the goal is to maximize $I(\mathbf{x},\hat{\mathbf{x}})$ while respecting the power constraints.

1) Power Constraint on the Source: In that case we have a single power constraint on the power emitted by the source. It can be expressed as $\mathbb{E}[\|\mathbf{A_x x}\|_{l_2}^2] \leq P_T$, where P_T is the maximum power available at the source. This power constraint can be written as

$$\mathrm{Tr}(\mathbf{A_x}\mathbf{A_x}^{\mathrm{H}}) \leq P_T. \tag{16}$$

Now by defining $\mathbf{R_M} = \mathbf{M_x}^{\mathrm{H}}(\mathbf{M}_{\boldsymbol\eta}\mathbf{A}_{\boldsymbol\eta}\mathbf{C}_{\boldsymbol\eta}\mathbf{A}_{\boldsymbol\eta}{}^{\mathrm{H}}\mathbf{M}_{\boldsymbol\eta}{}^{\mathrm{H}})^{-1}\mathbf{M_x}$ we can write the optimization problem we want to solve as

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}^{\mathrm{H}}\mathbf{R_M}\mathbf{A_x}) \\ \text{s.t.}\quad & \mathrm{Tr}(\mathbf{A_x}\mathbf{A_x}^{\mathrm{H}}) \leq P_T, \end{aligned} \tag{17}$$

which has exactly the form of the linear precoding problem of the single user MIMO channel [12].

Theorem 2: The optimal precoding matrix $\mathbf{A_x}^*$ for the problem (17) has the form

$$\mathbf{A_x}^* = \mathbf{V_M}\ \mathrm{diag}(\sqrt{p}), \tag{18}$$

where $\mathbf{V_M} \in \mathbb{C}^{n\times n}$ is the right singular matrix of $\mathbf{R_M}$ and p is a power allocation vector calculated using waterfilling as

$$p_i = (\mu - \lambda_{M,i}^{-1})^+, \qquad 1 \leq i \leq n, \tag{19}$$

where $\lambda_{M,i}$ are the eigenvalues of $\mathbf{R_M}$ and μ is the water level chosen such that $\sum_{i=1}^n p_i = P_T$.

Proof: See [12]. ∎

2) Network with a Global Power Constraint: In this scenario we assume that the network as a whole has a global power constraint, i.e., all nodes share a joint power budget. Now this case is much less similar to the MIMO problem than the former one. The key component here is to recognize that the vector $\mathbf{s_N} = \mathbf{MAs}$ of size equal to the number of edges in the network contains the output signal at each edge of the network. Among all these components we want to constraint only outgoing edges from the source or from relay nodes. The power $\mathbf{p_G} \in \mathbb{R}^e$ on each edge of the network can be written as

$$\mathbf{p_G} = \mathbb{E}[\mathrm{diag}(\mathbf{MAss}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{M}^{\mathrm{H}})]. \tag{20}$$

Unfortunately the information about which edge comes out of which node is not included in the matrix $\mathbf{F}$ as it only represents connexions from edge to edge. Therefore we need to introduce a new matrix $\mathbf{S}$ which is a selection matrix keeping only the diagonal entries of $\mathbf{F}$ corresponding to edges going out of a source or a relay. The matrix $\mathbf{S} \in \mathbb{C}^{e\times e}$ is a diagonal matrix defined as follow

$$s_{i,i} = \begin{cases} 1 & \text{if edge } i \text{ goes out of the source or a relay} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Note that we could have skipped the multiplicative node in the network model and have introduced the channel gain directly in $\mathbf{A}$. The big advantage of not having done that is visible now. The output signal of relays before being amplified by the channel gain (i.e., relevant for the power constraints) is readily available from $\mathbf{p_N}$. Finally we express the true power constraint of interest as

$$\mathrm{Tr}(\mathbb{E}[\mathbf{SMAss}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{M}^{\mathrm{H}}]) \leq P_T. \tag{22}$$

The problem consisting in minimizing the MSE with a global power constraint for the network can be formulated as

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}^{\mathrm{H}}\mathbf{R_M}\mathbf{A_x}) \\ \text{s.t.}\quad & \mathrm{Tr}(\mathbb{E}[\mathbf{SMAss}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{M}^{\mathrm{H}}]) \leq P_T. \end{aligned} \tag{23}$$

Theorem 3: The optimal precoding matrix $\mathbf{A_x}^*$ for the problem (17) has the form

$$\mathbf{A_x}^* = \mathbf{U_M}\mathbf{\Lambda_M}^{-1/2}\mathbf{V_G}\ \mathrm{diag}(\sqrt{p}), \tag{24}$$

where $\mathbf{V_G} \in \mathbb{C}^{n\times n}$ is the right singular matrix of $\mathbf{R_G}$, which is itself a $n\times n$ matrix depending on $\mathbf{R_M}$ ($\mathbf{R_G}$ is defined explicitly latter in the proof), $\mathbf{U_M}$ and $\mathbf{\Lambda_M}^{-1/2}$ are respectively a $n\times n$ unitary matrix and a $n\times n$ diagonal matrix both defined latter in the proof and p is a power allocation vector calculated using waterfilling as

$$p_i = (\mu - \lambda_{G,i}^{-1})^+, \qquad 1 \leq i \leq n, \tag{25}$$

where $\lambda_{G,i}$ are the eigenvalues of $\mathbf{R_G}$ and μ is the water level chosen such that $\sum_{i=1}^n p_i = P_T - P_N$ with P_N an amount of power depending on the network structure.

Proof: In problem (23) the constraint does not let $\mathbf{A_x}$ appear. Therefore we rewrite $\mathbf{M}$ as a block matrix with the following structure

$$\mathbf{M} = \begin{bmatrix} \mathbf{L_1} & \mathbf{0} \\ \mathbf{M_{LL}} & \mathbf{L_2} \end{bmatrix}, \tag{26}$$

where $\mathbf{L_1}$ is a lower diagonal $n\times n$ matrix, $\mathbf{L_2}$ a lower diagonal $(e-n)\times(e-n)$ matrix and $\mathbf{M_{LL}}$ a $(e-n)\times n$ matrix. We then derive $\mathbb{E}[\mathbf{MAss}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{M}^{\mathrm{H}}]$ as following

$$\mathbb{E}[\mathbf{MA}ss^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{M}^{\mathrm{H}}] = \begin{bmatrix} \mathbf{L_1A_xA_x}^{\mathrm{H}}\mathbf{L_1}^{\mathrm{H}} & \mathbf{L_1A_xA_x}^{\mathrm{H}}\mathbf{M_{LL}}^{\mathrm{H}} \\ \mathbf{M_{LL}A_xA_x}^{\mathrm{H}}\mathbf{L_1}^{\mathrm{H}} & \mathbf{M_{LL}A_xA_x}^{\mathrm{H}}\mathbf{M_{LL}}^{\mathrm{H}} \\ & +\mathbf{L_2A}_{\boldsymbol\eta}\mathbf{C}_{\boldsymbol\eta}\mathbf{A}_{\boldsymbol\eta}{}^{\mathrm{H}}\mathbf{L_2}^{\mathrm{H}} \end{bmatrix}. \tag{27}$$

Because we can always numerate the outgoing edges from the source first (i.e., they have $1,\ldots,n$ as indices) $\mathbf{L_1} = \mathbf{I}_n$ and by rewriting $\mathbf{S}$ as

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{D_S} \end{bmatrix}, \tag{28}$$

we can rewrite constraint (22) as

$$\mathrm{Tr}(\mathbf{A_xA_x}^{\mathrm{H}} + \mathbf{D_S}(\mathbf{M_{LL}A_xA_x}^{\mathrm{H}}\mathbf{M_{LL}}^{\mathrm{H}} + \mathbf{L_2A}_{\boldsymbol\eta}\mathbf{C}_{\boldsymbol\eta}\mathbf{A}_{\boldsymbol\eta}{}^{\mathrm{H}}\mathbf{L_2}^{\mathrm{H}})) \leq P_T. \tag{29}$$

The problem (23) can be reformulated as

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}^{\mathrm{H}}\mathbf{R_M}\mathbf{A_x}) \\ \text{s.t.}\quad & \mathrm{Tr}(\mathbf{A_xA_x}^{\mathrm{H}} + \mathbf{D_S M_{LL}A_xA_x}^{\mathrm{H}}\mathbf{M_{LL}}^{\mathrm{H}} + \\ & \mathbf{D_S L_2A}_{\boldsymbol\eta}\mathbf{C}_{\boldsymbol\eta}\mathbf{A}_{\boldsymbol\eta}{}^{\mathrm{H}}\mathbf{L_2}^{\mathrm{H}}) \leq P_T. \end{aligned} \tag{30}$$

By calling $P_N = \text{Tr}(\mathbf{D_S L_2 A_\eta C_\eta A_\eta}^{\text{H}}\mathbf{L_2}^{\text{H}})$ and manipulating the expression in the trace, the problem simplifies to

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}^{\text{H}}\mathbf{R_M A_x}) \\ \text{s.t.}\quad & \text{Tr}(\mathbf{A_x}^{\text{H}}(\mathbf{I}_n + \mathbf{M_{LL}}^{\text{H}}\mathbf{D_S M_{LL}})\mathbf{A_x}) \le P_T - P_N. \end{aligned} \tag{31}$$

Note that if we have $P_N > P_T$, it means that any amount of power used by the source will violate the global network constraint. The matrix $\mathbf{I}_n + \mathbf{M_{LL}}^{\text{H}}\mathbf{D_S M_{LL}}$ is symmetric, full ranked and can be written as

$$\begin{aligned} \mathbf{I}_n + \mathbf{M_{LL}}^{\text{H}}\mathbf{D_S M_{LL}} &= \mathbf{U_M \Lambda_M U_M}^{\text{H}} \\ &= \mathbf{U_M \Lambda_M}^{1/2}\mathbf{\Lambda_M}^{1/2}\mathbf{U_M}^{\text{H}}, \end{aligned} \tag{32}$$

where $\mathbf{U_M} \in \mathbb{C}^{n\times n}$ is a unitary matrix and $\mathbf{\Lambda_M} \in \mathbb{C}^{n\times n}$ is a diagonal matrix containing the eigenvalues of $\mathbf{I}_n + \mathbf{M_{LL}}^{\text{H}}\mathbf{D_S M_{LL}}$. Next we proceed to the following variable change

$$\mathbf{A_x}' = \mathbf{\Lambda_M}^{1/2}\mathbf{U_M}^{\text{H}}\mathbf{A_x} \tag{33}$$

where $\mathbf{A_x}'$ is a matrix of size $n \times n$. It remains to plug (33) in (31) to get the following transformed objective function

$$\log\det(\mathbf{I}_n + \mathbf{A_x}'^{\text{H}}\mathbf{\Lambda_M}^{-1/2}\mathbf{U_M}^{\text{H}}\mathbf{R_M U_M \Lambda_M}^{-1/2}\mathbf{A_x}'). \tag{34}$$

We define $\mathbf{R_G} = \mathbf{\Lambda_M}^{-1/2}\mathbf{U_M}^{\text{H}}\mathbf{R_M U_M \Lambda_M}^{-1/2}$ with $\mathbf{R_G}$ a $n \times n$ matrix and get the simplified problem

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}'^{\text{H}}\mathbf{R_G A_x}') \\ \text{s.t.}\quad & \text{Tr}(\mathbf{A_x}'^{\text{H}}\mathbf{A_x}') \le P_T - P_N. \end{aligned} \tag{35}$$

Now we got to the exact same form as in (17), which concludes the proof. ■

3) Relays with Individual Power Constraints: In this scenario we assume that all nodes (source and relays) have the same individual power available P_T. We have shown in the previous section that $\mathbb{E}[(\mathbf{SMAss}^{\text{H}}\mathbf{A}^{\text{H}}\mathbf{M}^{\text{H}})]$ represents the power on outgoing edges from the source and from the relays. In order to express individual power constraints for each node we need to group the edges belonging to the same node. We define c matrices $\mathbf{S_i}$ with $i = 1, \dots, c$ and c is the number of relays in the network plus one (for the source). We give index one to the source and indices between 2 and c to the relays. We define $\mathbf{S_i}$ as a $e \times e$ diagonal matrix of the form

$$[s_i]_{j,j} = \begin{cases} 1 & \text{if edge j goes out node i} \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

In other words by multiplying $\mathbf{S_i}$ with $\mathbf{MAss}^{\text{H}}\mathbf{A}^{\text{H}}\mathbf{M}^{\text{H}}$, it remains a $e \times e$ matrix with only the power on edges outgoing from node i on its diagonal. Therefore the network power constraint can be written as

$$\mathbb{E}[\text{Tr}(\mathbf{S_i MAss}^{\text{H}}\mathbf{A}^{\text{H}}\mathbf{M}^{\text{H}})] \le P_{Ti}, \qquad \forall i = 1, \dots, c. \tag{37}$$

where P_{Ti} is the power available at node i. The problem consisting in maximizing $I(\mathbf{x}, \hat{\mathbf{x}})$ with individual power constraints for each node can be formulated as

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}^{\text{H}}\mathbf{R_M A_x}) \\ \text{s.t.}\quad & \mathbb{E}[\text{Tr}(\mathbf{S_1 MAss}^{\text{H}}\mathbf{A}^{\text{H}}\mathbf{M}^{\text{H}})] \le P_{T1} \\ & \mathbb{E}[\text{Tr}(\mathbf{S_2 MAss}^{\text{H}}\mathbf{A}^{\text{H}}\mathbf{M}^{\text{H}})] \le P_{T2} \\ & \vdots \\ & \mathbb{E}[\text{Tr}(\mathbf{S_c MAss}^{\text{H}}\mathbf{A}^{\text{H}}\mathbf{M}^{\text{H}})] \le P_{Tc}. \end{aligned} \tag{38}$$

For each matrix $\mathbf{S_i}$, with $i > 1$, we will write the matrix $\mathbf{M}$ with a specific block structure

$$\mathbf{M} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M_{1,1}}_i & \mathbf{L_1}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{M_{2,1}}_i & \mathbf{M_{2,2}}_i & \mathbf{L_2}_i & \mathbf{0} \\ \mathbf{M_{3,1}}_i & \mathbf{M_{3,2}}_i & \mathbf{M_{3,3}}_i & \mathbf{L_3}_i \end{bmatrix}. \tag{39}$$

If we call e_i the number of edges between the last outgoing edge from the source and the first outgoing edge from node i and n_i the number of outgoing edges from node i, we have $\mathbf{M_{1,1}}_i \in \mathbb{C}^{e_i \times n}$, $\mathbf{M_{2,1}}_i \in \mathbb{C}^{n_i \times n}$, $\mathbf{M_{2,2}}_i \in \mathbb{C}^{n_i \times e_i}$, $\mathbf{M_{3,1}}_i \in \mathbb{C}^{(e-n-e_i-n_i)\times n}$, $\mathbf{M_{3,2}}_i \in \mathbb{C}^{(e-n-e_i-n_i)\times e_i}$, $\mathbf{M_{3,3}}_i \in \mathbb{C}^{(e-n-e_i-n_i)\times n_i}$, $\mathbf{L_1}_i \in \mathbb{C}^{e_i \times e_i}$, $\mathbf{L_2}_i \in \mathbb{C}^{n_i \times n_i}$ and $\mathbf{L_3}_i \in \mathbb{C}^{(e-n-e_i-n_i)\times(e-n-e_i-n_i)}$. We also rewrite $\mathbf{A}$ for each $i > 1$ as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_x} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\boldsymbol{\eta_1}i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\boldsymbol{\eta_2}i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\boldsymbol{\eta_3}i} \end{bmatrix}, \tag{40}$$

with $\mathbf{A}_{\boldsymbol{\eta_1}i} \in \mathbb{C}^{e_i \times e_i}$, $\mathbf{A}_{\boldsymbol{\eta_2}i} \in \mathbb{C}^{n_i \times n_i}$ and $\mathbf{A}_{\boldsymbol{\eta_3}i} \in \mathbb{C}^{(e-n-e_i-n_i)\times(e-n-e_i-n_i)}$. After some calculations and seeing that $\mathbf{S_i}$ has the form

$$\mathbf{S_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_i} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

we can transform the constraint (37), for each $i > 1$, into

$$\begin{aligned} \text{Tr}(&\mathbf{M_{2,1}}_i\mathbf{A_x A_x}^{\text{H}}\mathbf{M_{2,1}}_i^{\text{H}} + \mathbf{M_{2,2}}_i\mathbf{A}_{\boldsymbol{\eta_1}i}\mathbf{A}_{\boldsymbol{\eta_1}i}^{\text{H}}\mathbf{M_{2,2}}_i^{\text{H}} + \\ &\mathbf{L_2}_i\mathbf{A}_{\boldsymbol{\eta_2}i}\mathbf{A}_{\boldsymbol{\eta_2}i}^{\text{H}}\mathbf{L_2}_i^{\text{H}}) \le P_{Ti}, \qquad \forall i = 2, \dots, c. \end{aligned} \tag{42}$$

For $i = 1$ (for the source), the constraint (37) remains

$$\text{Tr}(\mathbf{A_x A_x}^{\text{H}}) \le P_{T1}. \tag{43}$$

By calling $P_{Ni} = \text{Tr}(\mathbf{M_{2,2}}_i\mathbf{A}_{\boldsymbol{\eta_1}i}\mathbf{A}_{\boldsymbol{\eta_1}i}^{\text{H}}\mathbf{M_{2,2}}_i^{\text{H}} + \mathbf{L_2}_i\mathbf{A}_{\boldsymbol{\eta_2}i}\mathbf{A}_{\boldsymbol{\eta_2}i}^{\text{H}}\mathbf{L_2}_i^{\text{H}})$ the problem (38) transforms to

$$\begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + \mathbf{A_x}^{\text{H}}\mathbf{R_M A_x}) \\ \text{s.t.}\quad & \text{Tr}(\mathbf{A_x}^{\text{H}}\mathbf{A_x}) \le P_{T1} \\ & \text{Tr}(\mathbf{A_x}^{\text{H}}\mathbf{M_{2,1}}_i^{\text{H}}\mathbf{M_{2,1}}_i\mathbf{A_x}) \le P_{Ti} - P_{Ni}, \\ & i = 2, \dots, c. \end{aligned} \tag{44}$$

In that case the matrix $\mathbf{M_{2,1}}_i^{\text{H}}\mathbf{M_{2,1}}_i$ has rank n_i, i.e., the number of outgoing edges from node i. This number is in general smaller than n. It means that the source could theoretically put all its power on the eigenmodes corresponding to zero eigenvalues of $\mathbf{M_{2,1}}_i^{\text{H}}\mathbf{M_{2,1}}_i$ and never violates its power constraint. However since there exists no unconstrained path between the source and the sink (there is at least one relay between the source and the sink and the source is itself power constrained), the power will be limited somewhere else in the network. The low rank nature of ${\mathbf{M_{2,1}}_i}^{\mathrm{H}}\mathbf{M_{2,1}}_i$ only expresses the fact that not all paths go through a specific node. The consequence is that we can rewrite the constraints by suppressing the unused eigenmodes of ${\mathbf{M_{2,1}}_i}^{\mathrm{H}}\mathbf{M_{2,1}}_i$, by writing

$$ {\mathbf{M_{2,1}}_i}^{\mathrm{H}}\mathbf{M_{2,1}}_i = \mathbf{U}_{\mathbf{M_{2,1}}_i}\begin{bmatrix}\boldsymbol{\Lambda}_{M_{2,1_i}} & \\ & \mathbf{0}\end{bmatrix}{\mathbf{U}_{\mathbf{M_{2,1}}_i}}^{\mathrm{H}}, \tag{45} $$

where $\mathbf{U}_{\mathbf{M_{2,1}}_i}$ is a unitary matrix of size $n \times n$ and $\boldsymbol{\Lambda}_{M_{2,1_i}}$ is a $n_i \times n_i$ diagonal matrix containing the nonzero eigenvalues of ${\mathbf{M_{2,1}}_i}^{\mathrm{H}}\mathbf{M_{2,1}}_i$. By keeping only the n_i rows of $\mathbf{U}_{\mathbf{M_{2,1}}_i}$ corresponding to $\boldsymbol{\Lambda}_{M_{2,1_i}}$ and calling $\mathbf{R}_{\mathbf{M}i} = \mathbf{U}'_{\mathbf{M_{2,1}}_i}\boldsymbol{\Lambda}_{M_{2,1_i}}{\mathbf{U}'_{\mathbf{M_{2,1}}_i}}^{\mathrm{H}}$ where $\mathbf{U}'_{\mathbf{M_{2,1}}_i}$ is a unitary $n_i \times n$ matrix, we transform problem (44) into

$$ \begin{aligned} \max_{\mathbf{A_x}}.\quad & \log\det(\mathbf{I}_n + {\mathbf{A_x}}^{\mathrm{H}}\mathbf{R_M}\mathbf{A_x}) \\ \text{s.t.}\quad & \mathrm{Tr}({\mathbf{A_x}}^{\mathrm{H}}\mathbf{A_x}) \le P_{T1} \\ & \mathrm{Tr}({\mathbf{A_x}}^{\mathrm{H}}\mathbf{R}_{\mathbf{M}i}\mathbf{A_x}) \le P_{Ti} - P_{Ni}, \\ & i = 2, \ldots, c. \end{aligned} \tag{46} $$

The problem formulation is finished, we see c constraints in potentially c different basis. We cannot use a change of variable to bring them all in the same basis so the optimal solution will be trade-off between those different basis. This however a complicated problem.

A greedy suboptimal algorithm consists in solving the simple problem (17) with a power constraint on the source and increase the power using waterfilling until it violates some relay constraints. When some constraints are violated, we take out of the waterfilling the power components that influence the violated constraints and continue the waterfilling on the remaining power components. It will be part of future works to find more efficient algorithms to solve (46).

VI. Numerical Results

In this section we present few simulation results from the network depicted in Figure 2. In order to get results that are interpretable we take $h_2 = h_1/2$ (i.e., the channel gain of the right outgoing edge from the source is twice weaker than the left one). All h_i (except h_2), as well as all coefficients $\tilde{f}_{ij}$ chosen by the relays are zero-mean Gaussian with variance one. The noise covariance matrix is such that $\mathbf{C}_{\boldsymbol{\eta}} = \mathbf{I}_N$, i.e., the noise is white. We first concentrate on the network with a power constraint on the source only. In Figure 4, we show the mutual information between the source signal and the sink signal, depending on the SNR, for the cases that 1) both relays are active, 2) the left relay is inactive 3) the right relay is inactive. In Figure 5, we show the strategy chosen by the source in the case that the two relays are active (i.e., the power that goes in the left and right relays). We can see as expected that because h_2 is worse than h_1, the performance of the network is better when the only left relay is active than when only the right relay is active. When both relays are active, the performance at low SNR is about the same as when only the left relay is active. In that case the source can spend all its power on the left side since the other side has a poor quality. When the SNR grows, the right side starts to be interesting for the source and we see from Figure 5 that the power distribution ratio tends to one. In other words the source distributes its power more and more fairly among sides.

It is difficult to compare networks with different power constraints as described in Section V. We simulated the network with a power constraint on the source only and then with a global power constraint on the network. In the first

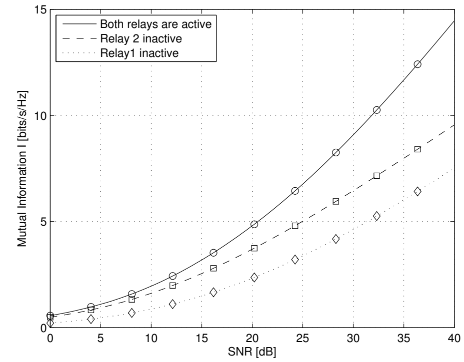

Fig. 4: Mutual information for three network configurations.

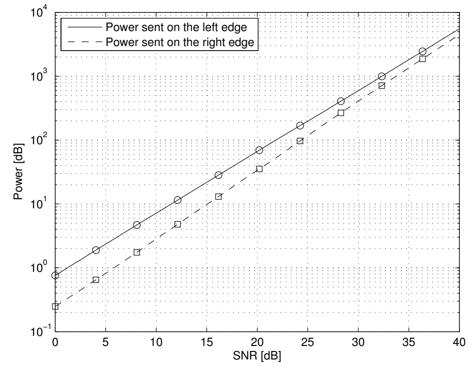

Fig. 5: Power distribution from the source among left and right edges.

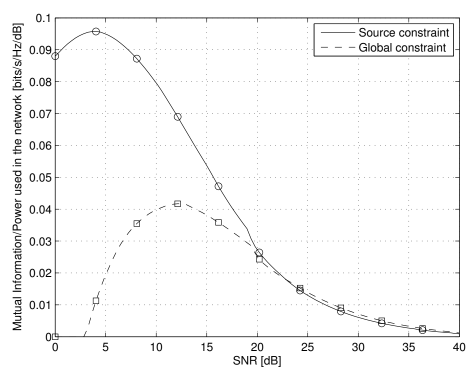

Fig. 6: Efficiency comparison of source constrained and globally constrained networks.

case the source has P_T power available and in the second case the whole network has P_T power available. To still be able to compare both strategies we plot the mutual information divided by the power consumed in the whole network in Figure 6. At low SNR, the source and the relays in the network with a global constraint have so few power available that the efficiency stays close to zero. At high SNR however, the power can be used very efficiently, all on the left side, which compensates the difference in available power and the efficiency of both network converges.

VII. Conclusion

In this work we presented a new framework for amplify-and-forward Gaussian relay networks. By modeling the network as a matrix and considering the relaying strategy as fixed we have drawn a strong connexion between relay networks and MIMO systems. Using our model we designed optimal filter for the sink and optimal precoder for the source for different power constraints in the network. We now envisage to develop algorithms for networks with several sources and sinks as well as more advanced methods for networks with individual power constraints on each node.

Appendix
System Model for Multi-Source Multi-Sink Relay Networks

In this section we extend our relay network model to represent multi-source multi-sink networks. We assume a network with N_T sources and N_R sinks. Each source j transmits a vector $\tilde{\mathbf{x}}_j \in \mathbb{C}^{n'}$. We define $\mathbf{x} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{N_T}) \in \mathbb{C}^n$ as the global transmitted vector with $n = N_T n'$. Here also we assume that $\mathbf{x}$ is zero-mean, normalized and uncorrelated such that $\mathbf{x} \sim \mathcal{N}_{\mathbb{C}}(0, \mathbf{I}_n)$. Each sink i is required to recover the data from all sources, i.e., $\mathbf{x}$, and actually receives a vector $\hat{\mathbf{x}}_i \in \mathbb{C}^n$. Typically we want to maximize the mutual information between $\mathbf{x}$ and $\hat{\mathbf{x}}_i$ for $i = 1, \ldots, N_R$.

The vector $\mathbf{s} \in \mathbb{C}^{n+N}$ remains unchanged and is defined as

$$\mathbf{s} = \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{pmatrix}. \tag{47}$$

The structure of the matrix $\mathbf{A} \in \mathbb{C}^{e \times (n+N)}$ stays the same, i.e.,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\boldsymbol{\eta}} \end{bmatrix}, \tag{48}$$

but $\mathbf{A_x} \in \mathbb{C}^{n \times n}$ is now a block diagonal matrix of the format

$$\mathbf{A_x} = \begin{bmatrix} \mathbf{A_{x}}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A_{x}}_{N_T} \end{bmatrix}, \tag{49}$$

where $\mathbf{A_{x}}_j \in \mathbb{C}^{n' \times n'}$ represents the amplification factors applied by the source j to $\tilde{\mathbf{x}}_j$. The network matrix $\mathbf{M} \in \mathbb{C}^{e \times e}$ can, in multi-source multi-sink network, be detailed as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M_L}_1 & \ldots & \mathbf{M_L}_{N_T} & \mathbf{M_R} \\ \mathbf{M_x}_{1,1} & \ldots & \mathbf{M_x}_{1,N_T} & \mathbf{M}_{\boldsymbol{\eta}1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{M_x}_{N_R,1} & \ldots & \mathbf{M_x}_{N_R,N_T} & \mathbf{M}_{\boldsymbol{\eta}N_R} \end{bmatrix}, \tag{50}$$

with $\mathbf{M_x}_{i,j} \in \mathbb{C}^{n \times n'}$ is a matrix containing the amplification factors experienced by $\tilde{\mathbf{x}}_j$ on the way to the sink i, $\mathbf{M_L}_j \in \mathbb{C}^{(e-N_R n) \times n'}$ contains amplification factors of $\tilde{\mathbf{x}}_j$ to the relays, $\mathbf{M_R} \in \mathbb{C}^{(e-N_R n) \times N}$ contains amplification factors of the noise to the relays and finally $\mathbf{M}_{\boldsymbol{\eta}i} \in \mathbb{C}^{n \times N}$ contains amplification factors of the noise to the sink i. Finally the filtering matrix $\mathbf{B}$ is now composed of one filtering matrix $\mathbf{B}_i \in \mathbb{C}^{e \times n}$ per sink and has the form $\mathbf{B} = [\mathbf{B}_1, \ldots, \mathbf{B}_{N_R}]$, which can be further expressed as

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{(e-N_R n) \times n} & \mathbf{0}_{(e-N_R n) \times n} & \mathbf{0}_{(e-N_R n) \times n} \\ \mathbf{B_x}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{B_x}_{N_R} \end{bmatrix}, \tag{51}$$

where $\mathbf{B_x}_i \in \mathbb{C}^{n \times n}$ is a filter applied by the sink i to the received signal.

Similar to (6) we can express the received signal at the sink i as

$$\hat{\mathbf{x}}_i = \mathbf{B}_i^{\mathrm{H}} \mathbf{M} \mathbf{A} \mathbf{s}. \tag{52}$$

Further this expression can be detailed by using (48), (50) and (51) as follow

$$\hat{\mathbf{x}}_i = \mathbf{B_x}_i^{\mathrm{H}} (\mathbf{M_x}_i \mathbf{A_x} \mathbf{x} + \mathbf{M}_{\boldsymbol{\eta}i} \mathbf{A}_{\boldsymbol{\eta}} \boldsymbol{\eta}), \tag{53}$$

where $\mathbf{M_x}_i = [\mathbf{M_x}_{i,1}, \ldots, \mathbf{M_x}_{i,N_T}]$.

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