Degrees of Freedom of the MIMO 3-Way Channel

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***Abstract*—In the present paper, we consider the Degrees-of-Freedom (DoF) of a multiple-input multiple-output (MIMO) 3-way channel with an arbitrary number of antennas at each user. This channel provides a particular extension of the two-way channel to three users. Therein, three users exchange six messages in total, i. e., there is one message from each user to each of the two other users. We derive upper bounds on the DoF of the channel and show that those are achievable by MIMO interference alignment (IA) and zero-forcing beam-forming. We show that the network has a number of $2M_2$ DoF where M_j represents the number of antennas at user j, and $M_1 \geq M_2 \geq M_3$.**

I. Introduction

The impact of interference is a natural impairment in wireless multi-user communication networks. Since the exact characterization of capacity for multiple interfering users is a very challenging task, approximate measures of the channel capacity are used to study its asymptotic behaviour. A capacity approximation which becomes accurate in the high signal-to-noise ratio (SNR) regime is termed the degrees-of-freedom (DoF) [1], which is also known as the capacity pre-log factor or multiplexing gain.

As introduced by the seminal works [2] and [3], the concept of interference alignment (IA) is shown to be a key method to achieve the upper bounds on the DoF in the presence of multi-user interference. Quite extensive work on the DoF for various multi-user interference networks has already been accomplished. A particular object of interest concerns the application of IA in MIMO channels with constant channel coefficients. For instance, the DoF of the 2-user MIMO interference channel using zero-forcing are provided in [1], the DoF and the DoF region of the 2-user MIMO X-channel are considered in [4] and [5], respectively, where IA was used.

In this paper, we apply IA to a multi-way communications scenario, i.e., a scenario where a user transmits some data to the other users and simultaneously receives some data from the other users. In particular, we consider a 3-way channel (Fig. 1), which can be considered as an extension of Shannon's two-way channel [6] to three users. Note that this mode of communications (multi-way) is natural since a significant part of our daily communications is two-way or multi-way (video conferences for instance). This mode also applies to the current hot-topic of device-to-device communications [7]–[9].

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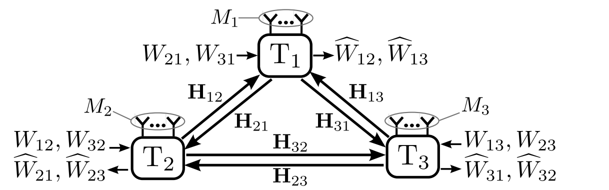

Fig. 1. The MIMO 3-way channel (or Δ-channel) with M_i transmit and M_i receive antennas at each user T_i, with $i = 1, 2, 3$.

Multi-way communications has been considered earlier in the context of multi-way relay channels. For instance, the DoF of the MIMO 3-way relay channel, known as the Y-channel, have been studied in [10] and [11]. In this paper, we consider multi-way communications without a dedicated relay node (contrary to [10], [11]). We study a MIMO 3-way channel, where three full-duplex users intend to exchange messages with each other directly as depicted in Fig. 1. The single-input single-output (SISO) variant of the 3-way channel has been studied in [12], where the sum-capacity was characterized within 2 bits. The result of [12] states that the sum-capacity can be approached by letting the two strongest users communicate while leaving the third one silent. Note that the 3-way channel can be obtained from the 3-user X-network [13] with transmitters Tx_i and receivers Rx_i, $i = 1, 2, 3$, by using the following operations: remove the message from transmitter Tx_i to receiver Rx_i (setting its rate to zero) and provide a noiseless instantaneous cooperation channel between Tx_i and Rx_i. This is the main difference between the 3-way channel and the 3-user X-network. Another difference is that [13] considers time-varying MIMO channels, while we consider constant MIMO channels.

Contributions. In the present work, we study the DoF of the MIMO 3-way channel with constant channel coefficients and with an arbitrary number of M_i transmit antennas at each transceiver and the same number of M_i receive antennas. We derive cut-set and genie-aided upper bounds and obtain an upper bound on the sum-DoF of the channel. We also propose a MIMO IA and zero-forcing scheme to show that the derived sum-DoF upper bound is achievable. We observe that the sum-DoF is limited by the strongest channel (the one with the largest rank), and therefore, that the sum-DoF is achievable by letting the two strongest users communicate similar to the SISO case [12]. Since this approach does not serve all users, we propose an alternative scheme which also achieves the sum-DoF upper bound while serving all users. The achievable DoF of this alternative scheme is expressed as a linear program which can be solved by using the simplex method.

Organization. The system model of the MIMO 3-way channel is provided in Section II and the main result is stated in Section III. In Section IV, the upper bounds on the DoF are derived. The IA based transmission scheme is described in Section V, achieving the sum-DoF of the channel.

Notation. We denote matrices by boldface upper case letters, e. g., $\boldsymbol{A}$, and vectors by boldface lower case letters, e. g., $\boldsymbol{a}$. $\boldsymbol{a}^N$ denotes the length-N sequence $(\boldsymbol{a}(1),\cdots,\boldsymbol{a}(N))$. $\boldsymbol{A}^\mathsf{T}$ and $\boldsymbol{A}^\dagger$ denote the transposed matrix of $\boldsymbol{A}$ and its left Moore-Penrose pseudo-inverse. $\mathrm{span}(\boldsymbol{A})$, $\dim(\boldsymbol{A})$ and $\mathrm{null}(\boldsymbol{A})$ denote the column span, the dimension of the column space, and the null space of a matrix $\boldsymbol{A}$, respectively. An $n\times n$ identity matrix is denoted by $\boldsymbol{I}_n$ and an $a\times b$ zero matrix by $\mathbf{0}_{a\times b}$. Furthermore, let $(a)^+ = \max\{0,a\}$, for $a\in\mathbb{R}$.

II. System Model

The 3-way channel comprises three full-duplex[1] users T_i with user indices i in the set $\mathcal{K}=\{1,2,3\}$. A message from T_i to T_j is denoted by W_{ji} and has rate R_{ji} for $i\neq j\in\mathcal{K}$. Each user T_i desires to communicate a message to T_j and another message to T_k, for distinct i,j,k. A user T_i is equipped with an arbitrary number of antennas $M_i\in\mathbb{N}$, where the number of transmit and receive antennas is assumed to be equal. We may assume w. l. o. g. that the number of antennas is ordered among the three users by:

$$M_1\geq M_2\geq M_3. \tag{1}$$

The signal transmitted at time-instant n from T_i is a vector $\boldsymbol{x}_i(n)\in\mathbb{C}^{M_i\times 1}$, satisfying a power constraint P. The channel matrix for the MIMO channel from T_i to T_j is denoted $\boldsymbol{H}_{ji}\in\mathbb{C}^{M_j\times M_i}$. These random channel matrices are generated i.i.d from a continuous probability distribution and are assumed to be constant throughout the whole duration of the transmission. The received signal at T_j is a vector $\boldsymbol{y}_j(n)\in\mathbb{C}^{M_j\times 1}$. $\boldsymbol{y}_j(n)$ is a superposition of the transmitted signals from T_i and T_k, weighted by $\boldsymbol{H}_{ji},\boldsymbol{H}_{jk}$, respectively, and of i.i.d. complex additive white Gaussian noise $\boldsymbol{z}_j\sim\mathcal{CN}(\mathbf{0}_{M_j\times 1},\boldsymbol{I}_{M_j})$:

$$\boldsymbol{y}_j(n)=\boldsymbol{H}_{ji}\boldsymbol{x}_i(n)+\boldsymbol{H}_{jk}\boldsymbol{x}_k(n)+\boldsymbol{z}_j(n), \tag{2}$$

for distinct $i,j,k\in\mathcal{K}$. After receiving $\boldsymbol{y}_j(n)$, T_j constructs $\boldsymbol{x}_j(n+1)$ as:

$$\boldsymbol{x}_j(n+1)=\mathcal{E}_{j,n}(W_{ij},W_{kj},\boldsymbol{y}_j^n), \tag{3}$$

where $\mathcal{E}_{j,n}$ is the encoding function of T_j at time-instant n, and sends $\boldsymbol{x}_j(n+1)$ in the next transmission. After N transmissions, where N is the length of one transmission block (codeword), T_j decodes W_{ji} and W_{jk} as follows:

$$(W_{ji},W_{jk})=\mathcal{D}_j(W_{ij},W_{kj},\boldsymbol{y}_j^N), \tag{4}$$

where $\mathcal{D}_j$ is the decoding function of T_j.

[1]We assume perfect full-duplex operation, and hence, there is no residual loop-back self-interference at each receiving T_i.

All channel matrices are perfectly known at each user. In the rest of the paper, we will neglect the time-instant n for notational simplicity unless necessary.

Since the focus of the present paper is on the DoF [1] of the network, we define the DoF of a message W_{ji} by:

$$d_{ji}=\lim_{P\to\infty}\frac{R_{ji}}{\log(P)}. \tag{5}$$

Having defined the system model, we are ready to state the main results of the paper provided in the next section.

III. Main Result

The main result of the paper is a sum-DoF characterization for the MIMO 3-way channel as provided in the following theorem.

Theorem 1. *The DoF of the MIMO 3-way channel with M_i antennas at user T_i, and $M_1\geq M_2\geq M_3$, are given by:*

$$d_\Sigma=d_{12}+d_{21}+d_{13}+d_{31}+d_{23}+d_{32}=2M_2. \tag{6}$$

The converse of this theorem is provided in Section IV and the achievability in Section V. This theorem states that the sum-DoF in this case is given by twice the rank of the channel matrix between T_1 and T_2, which is the channel of largest rank. Therefore, this DoF is achievable by letting these two users communicate while leaving T_3 silent. Albeit this achieves $2M_2$ DoF, it completely excludes T_3 and it does not distribute the resources fairly between the three users. In Section V, we provide an alternative scheme which achieves the DoF, while maintaining non-zero DoF for all users.

IV. Converse

Cut-set bounds: We begin with considering the cut-set bounds for the MIMO 3-way channel:

$$d_{ji}+d_{ki}\leq\min\{M_i,M_j+M_k\}, \tag{7}$$
$$d_{ij}+d_{ik}\leq\min\{M_j+M_k,M_i\}. \tag{8}$$

The right-hand side of (7) is the rank of the MIMO channel between T_i and a receiver formed by enabling full cooperation between T_j and T_k, with channel matrix $[\boldsymbol{H}_{ji}^\mathsf{T}\boldsymbol{H}_{ki}^\mathsf{T}]^\mathsf{T}$. A similar interpretation holds for the second bound.

Similar to [10], the cut-set bounds provide bounds on the sum of the DoF of two messages at a time. However, using genie-aided arguments, it is possible to establish bounds on the sum-DoF of three messages, which are tighter than the cut-set bounds. The key idea is to allow some user to decode one more message, in addition to its two desired messages, by enhancing this user with some side-information.

Genie-aided bounds: Assume every node can obtain its dedicated messages with an arbitrary small probability of error. This means that T_2 for instance can decode its dedicated messages W_{21} and W_{23} reliably from its available information, i. e., from its own transmitted W_{12},W_{32}, and from its received signal $\boldsymbol{y}_2^N$. Now let us enhance T_2 by providing the message W_{31} as side-information. We also provide T_2 with the correction-noise signal:

$$\tilde{\boldsymbol{z}}_2^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{13}\boldsymbol{H}_{23}^{\dagger}\boldsymbol{z}_2^N, \tag{9}$$

as side-information[2].

At this point, T_2 knows W_{21} (decoded) and W_{31} (side-information). With W_{21}, W_{31}, T_2 can generate $\boldsymbol{x}_1(1)$. By subtracting $\boldsymbol{H}_{21}\boldsymbol{x}_1(1)$ from $\boldsymbol{y}_2(1)$, and multiplying the result with $\boldsymbol{H}_{23}^{\dagger}$, T_2 can recover a noisy observation of $\boldsymbol{x}_3(1)$ given by $\boldsymbol{x}_3(1) + \boldsymbol{H}_{23}^{\dagger}\boldsymbol{z}_2(1)$. Next, T_2 multiplies this noisy observation by $\boldsymbol{H}_{13}$, and adds $\boldsymbol{H}_{12}\boldsymbol{x}_2(1)$ and $\tilde{\boldsymbol{z}}_2(1)$ to it to obtain $\boldsymbol{y}_1(1)$. Thus, T_2 obtains the first instance of $\boldsymbol{y}_1^N$. Knowing $\boldsymbol{y}_1(1)$, W_{21} and W_{31}, T_2 can generate $\boldsymbol{x}_1(2)$ (cf. (3)). Using $\boldsymbol{x}_1(2)$ again with $\boldsymbol{y}_2(2)$, T_2 can generate $\boldsymbol{y}_1(2)$ and $\boldsymbol{x}_1(3)$. T_2 proceeds this way until all instances (up to the N-th instance) of $\boldsymbol{y}_1^N$ have been generated. Now, having $\boldsymbol{y}_1^N$, W_{21}, and W_{31}, i. e., the same information as T_1, T_2 can decode W_{13} (cf. (4)). Therefore, given W_{31} and $\tilde{\boldsymbol{z}}_2^N$ as side-information, T_2 can decode W_{21}, W_{23} and W_{13}. Hence, the DoF of these messages are almost surely upper bounded by:

$$d_{21} + d_{23} + d_{13} \le \operatorname{rank}([\boldsymbol{H}_{21}\, \boldsymbol{H}_{23}]) \tag{10}$$

$$= \min\{M_2, M_1 + M_3\} \overset{(1)}{=} M_2. \tag{11}$$

We can apply a similar approach to bound $d_{31} + d_{32} + d_{12}$ by M_2. However, in this case, we need to enhance T_3 with $M_2 - M_3$ antennas to make it as strong as T_2. The effective channel output at T_3 after this enhancement becomes:

$$\tilde{\boldsymbol{y}}_3(n) = \tilde{\boldsymbol{H}}_{31}\boldsymbol{x}_1(n) + \tilde{\boldsymbol{H}}_{32}\boldsymbol{x}_2(n) + \tilde{\boldsymbol{z}}_3(n), \tag{12}$$

for $n = 1, \cdots, N$, where $\tilde{\boldsymbol{H}}_{31}$ and $\tilde{\boldsymbol{H}}_{32}$ are $M_2 \times M_1$ and $M_2 \times M_2$ matrices with rank M_2, respectively, and $\tilde{\boldsymbol{z}}_3$ is a Gaussian noise vector with M_2 dimensions. T_3 can decode W_{31}, W_{32} having $\tilde{\boldsymbol{y}}_3^N$, W_{13}, W_{23}. By providing W_{21} and:

$$\tilde{\boldsymbol{z}}_3^N = \boldsymbol{z}_1^N - \boldsymbol{H}_{12}\tilde{\boldsymbol{H}}_{32}^{-1}\boldsymbol{z}_3^N \tag{13}$$

to the enhanced T_3 with M_2 antennas, it can generate $\boldsymbol{x}_1(1)$. We use analogous operations as applied for (11) to obtain $\boldsymbol{y}_1^N$ and to decode W_{12}. This leads to the upper bound:

$$d_{31} + d_{32} + d_{12} \le \operatorname{rank}([\tilde{\boldsymbol{H}}_{31}\, \tilde{\boldsymbol{H}}_{32}]) \tag{14}$$

$$= \min\{M_2, M_1 + M_2\} = M_2, \tag{15}$$

almost surely. Concluding the converse proof by combining (11) and (15) yields the sum-DoF upper bound of Theorem 1:

$$d_\Sigma = d_{12} + d_{21} + d_{13} + d_{31} + d_{23} + d_{32} \le 2M_2. \tag{16}$$

V. Achievability

To achieve the upper bound on the sum-DoF, we propose a beam-forming and zero-forcing scheme using MIMO interference alignment [10].

[2] $\boldsymbol{H}_{23}^{\dagger}$ exists since $\boldsymbol{H}_{23}$ is an $M_2 \times M_3$ matrix with $M_2 \ge M_3$.

A. *Pre-coding*

We consider the receive signal space at T_1 at first. Note that as T_2 and T_3 each have less antennas than T_1, they can not beam-form interference into the null space of T_1. Instead of zero-forcing beam-forming, we use IA. In order to minimize the number of dimensions spanned by the interference caused by T_2 and T_3 at T_1, we align the interference caused by the bidirectional communication between T_2 and T_3 (signals $\boldsymbol{u}_{32}$ and $\boldsymbol{u}_{23}$, respectively) in the intersection subspace of the spaces spanned by the columns of $\boldsymbol{H}_{12}$ and $\boldsymbol{H}_{13}$. From Lemma 2 as given in the appendix, the columns of $\boldsymbol{H}_{12}$ and $\boldsymbol{H}_{13}$ intersect in an $\tilde{M}_1$-dimensional subspace, where $\tilde{M}_1 = (M_2 + M_3 - M_1)^+$. To achieve this alignment, T_2 and T_3 pre-code the signal streams $\boldsymbol{u}_{32} \in \mathbb{C}^{\tilde{d}_{32}}$ and $\boldsymbol{u}_{23} \in \mathbb{C}^{\tilde{d}_{32}}$ with:

$$0 \le \tilde{d}_{32} = \tilde{d}_{23} \le \tilde{M}_1 \tag{17}$$

dimensions, into $\boldsymbol{V}_{32}\boldsymbol{u}_{32}$ and $\boldsymbol{V}_{23}\boldsymbol{u}_{23}$, respectively, where the beam-forming matrices $\boldsymbol{V}_{32} \in \mathbb{C}^{M_2 \times \tilde{d}_{32}}$ and $\boldsymbol{V}_{23} \in \mathbb{C}^{M_3 \times \tilde{d}_{32}}$ satisfy the following alignment at T_1:

$$\operatorname{span}(\boldsymbol{H}_{13}\boldsymbol{V}_{23}) = \operatorname{span}(\boldsymbol{H}_{12}\boldsymbol{V}_{32}). \tag{18}$$

This accounts for a total of $2\tilde{d}_{32}$ streams that can be exchanged by T_2 and T_3, while causing interference in only $\tilde{d}_{32}$ dimensions at T_1.

Now, we consider the receive signal space at T_2. As T_1 has more antennas than T_2, T_1 can send a signal $\bar{\boldsymbol{u}}_{31} \in \mathbb{C}^{\bar{d}_{31}}$ to T_3 in the null space of $\boldsymbol{H}_{21}$. The maximal number of such streams that can be beam-formed to this null space is bounded by $\min\{M_1 - M_2, M_3\}$. Thus, T_1 sends streams of:

$$0 \le \bar{d}_{31} \le \min\{M_1 - M_2, M_3\} \tag{19}$$

dimensions beam-formed into the null space of $\boldsymbol{H}_{21}$. To realize this, T_1 designs a zero-forcing beam-forming matrix $\bar{\boldsymbol{V}}_{31} \in \mathbb{C}^{M_1 \times \bar{d}_{31}}$ that satisfies:

$$\boldsymbol{H}_{21}\bar{\boldsymbol{V}}_{31} = \boldsymbol{0}_{M_2 \times \bar{d}_{31}}, \tag{20}$$

and pre-codes $\bar{\boldsymbol{u}}_{31}$ by $\bar{\boldsymbol{V}}_{31}\bar{\boldsymbol{u}}_{31}$. The remaining streams sent from T_1 to T_3 (if any) can be aligned to the streams sent from T_3 to T_1 within the receive signal space of T_2. This alignment is possible since the columns of $\boldsymbol{H}_{21}$ and $\boldsymbol{H}_{23}$ intersect in an M_3-dimensional subspace as given by Lemma 2. To this end, T_1 and T_3 construct $\tilde{\boldsymbol{V}}_{31}\tilde{\boldsymbol{u}}_{31}$ and $\boldsymbol{V}_{13}\boldsymbol{u}_{13}$, respectively, where $\tilde{\boldsymbol{u}}_{31} \in \mathbb{C}^{\tilde{d}_{31}}$ and $\boldsymbol{u}_{13} \in \mathbb{C}^{\tilde{d}_{13}}$ have:

$$0 \le \tilde{d}_{31} = \tilde{d}_{13} \le M_3 \tag{21}$$

dimensions, and where the beam-forming matrices defined by $\boldsymbol{V}_{13} \in \mathbb{C}^{M_3 \times \tilde{d}_{31}}$ and $\tilde{\boldsymbol{V}}_{31} \in \mathbb{C}^{M_1 \times \tilde{d}_{31}}$ satisfy:

$$\operatorname{span}(\boldsymbol{H}_{23}\boldsymbol{V}_{13}) = \operatorname{span}(\boldsymbol{H}_{21}\tilde{\boldsymbol{V}}_{31}). \tag{22}$$

The aligned interference of $\tilde{\boldsymbol{u}}_{31}$ and $\boldsymbol{u}_{13}$ occupies $\tilde{d}_{31} \le M_3$ dimensions at the receive signal space of T_2.

Considering the interference space at T_3, we see that T_3 has less antennas than T_1 and than T_2. Thus, T_1 beam-forms a signal $\bar{\boldsymbol{u}}_{21} \in \mathbb{C}^{\bar{d}_{21}}$ into the null space of $\boldsymbol{H}_{31}$ of size

$\min\{M_2, M_1 - M_3\}$, which requires:

$$0 \le \bar{d}_{21} \le \min\{M_2, M_1 - M_3\} \tag{23}$$

dimensions. This is done by designing a zero-forcing beam-forming matrix $\bar{\boldsymbol{V}}_{21} \in \mathbb{C}^{M_1 \times \bar{d}_{21}}$ such that:

$$\boldsymbol{H}_{31}\bar{\boldsymbol{V}}_{21} = \boldsymbol{0}_{M_3 \times \bar{d}_{21}}, \tag{24}$$

and by pre-coding $\bar{\boldsymbol{u}}_{21}$ with $\bar{\boldsymbol{V}}_{21}\bar{\boldsymbol{u}}_{21}$. Then, T_2 beam-forms $\bar{\boldsymbol{u}}_{12} \in \mathbb{C}^{\bar{d}_{12}}$ into the null space at T_3 of size $M_2 - M_3$, where:

$$0 \le \bar{d}_{12} \le M_2 - M_3. \tag{25}$$

To realize this, we design a zero-forcing beam-forming matrix $\bar{\boldsymbol{V}}_{12} \in \mathbb{C}^{M_2 \times \bar{d}_{12}}$ such that:

$$\boldsymbol{H}_{32}\bar{\boldsymbol{V}}_{12} = \boldsymbol{0}_{M_3 \times \bar{d}_{12}}, \tag{26}$$

and pre-code $\bar{\boldsymbol{u}}_{12}$ by $\bar{\boldsymbol{V}}_{12}\bar{\boldsymbol{u}}_{12}$. The remaining streams from T_1 to T_2 and vice versa (if any) are aligned at T_3. The spaces spanned by $\boldsymbol{H}_{31}$ and $\boldsymbol{H}_{32}$ intersect in M_3 dimensions as given by Lemma 2. We choose the beam-forming matrices $\tilde{\boldsymbol{V}}_{21} \in \mathbb{C}^{M_1 \times \tilde{d}_{21}}$ and $\tilde{\boldsymbol{V}}_{12} \in \mathbb{C}^{M_2 \times \tilde{d}_{21}}$ such that:

$$\mathrm{span}(\boldsymbol{H}_{32}\tilde{\boldsymbol{V}}_{12}) = \mathrm{span}(\boldsymbol{H}_{31}\tilde{\boldsymbol{V}}_{21}), \tag{27}$$

and use them to pre-code $\tilde{\boldsymbol{u}}_{21}$ and $\tilde{\boldsymbol{u}}_{12}$ with:

$$0 \le \tilde{d}_{21} = \tilde{d}_{12} \le M_3 \tag{28}$$

dimensions into $\tilde{\boldsymbol{V}}_{21}\tilde{\boldsymbol{u}}_{21}$ and $\tilde{\boldsymbol{V}}_{12}\tilde{\boldsymbol{u}}_{12}$.

Finally, the transmitters send the following signals:

$$\boldsymbol{x}_1 = [\tilde{\boldsymbol{V}}_{21}\ \bar{\boldsymbol{V}}_{21}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{21}\\ \bar{\boldsymbol{u}}_{21}\end{bmatrix} + [\tilde{\boldsymbol{V}}_{31}\ \bar{\boldsymbol{V}}_{31}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{31}\\ \bar{\boldsymbol{u}}_{31}\end{bmatrix}, \tag{29}$$

$$\boldsymbol{x}_2 = [\tilde{\boldsymbol{V}}_{12}\ \bar{\boldsymbol{V}}_{12}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{12}\\ \bar{\boldsymbol{u}}_{12}\end{bmatrix} + \boldsymbol{V}_{32}\boldsymbol{u}_{32}, \tag{30}$$

$$\boldsymbol{x}_3 = \boldsymbol{V}_{13}\boldsymbol{u}_{13} + \boldsymbol{V}_{23}\boldsymbol{u}_{23}. \tag{31}$$

In total, T_1 sends $d_{21} = \tilde{d}_{21} + \bar{d}_{21}$ and $d_{31} = \tilde{d}_{31} + \bar{d}_{31}$ streams to T_2 and T_3, respectively, T_2 sends $d_{12} = \tilde{d}_{12} + \bar{d}_{12}$ and $d_{32} = \tilde{d}_{32}$ streams to T_1 and T_3, respectively, and T_3 sends $d_{13} = \tilde{d}_{12}$ and $d_{23} = \tilde{d}_{23}$ streams to T_1 and T_2, respectively.

B. Post-coding

The received signal at T_1 can be written as:

$$\boldsymbol{y}_1 = \boldsymbol{H}_{12}[\tilde{\boldsymbol{V}}_{12}\ \bar{\boldsymbol{V}}_{12}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{12}\\ \bar{\boldsymbol{u}}_{12}\end{bmatrix} + [\boldsymbol{H}_{12}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{H}_{13}\boldsymbol{V}_{23}\boldsymbol{u}_{23}] + \boldsymbol{H}_{13}\boldsymbol{V}_{13}\boldsymbol{u}_{13} + \boldsymbol{z}_1. \tag{32}$$

The desired signals from T_2 occupy $\tilde{d}_{21} + \bar{d}_{21}$ dimensions. The aligned interference $\boldsymbol{H}_{12}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{H}_{13}\boldsymbol{V}_{23}\boldsymbol{u}_{23}$ occupies $\tilde{d}_{32}$ dimensions, and the desired signal from T_3 occupies $\tilde{d}_{13}$ dimensions. The desired signals can be resolved from the interference as long as they are linearly independent of the interference and also among each other. Namely, the columns of the following $M_1 \times (\tilde{d}_{12} + \bar{d}_{12} + \tilde{d}_{32} + \tilde{d}_{13})$ matrix must be linearly independent:

$$\left[\boldsymbol{H}_{12}\tilde{\boldsymbol{V}}_{12}\ \boldsymbol{H}_{12}\bar{\boldsymbol{V}}_{12}\ \boldsymbol{H}_{12}\boldsymbol{V}_{32}\ \boldsymbol{H}_{13}\boldsymbol{V}_{13}\right], \tag{33}$$

which requires:

$$0 \le \tilde{d}_{12} + \bar{d}_{12} + \tilde{d}_{32} + \tilde{d}_{13} \le M_1. \tag{34}$$

Under this condition, this linear independence can be guaranteed (almost surely) by designing $\bar{\boldsymbol{V}}_{12}$ according to (26), and choosing $\tilde{\boldsymbol{V}}_{12}$, $\boldsymbol{V}_{32}$, and $\boldsymbol{V}_{13}$ randomly.

Given this linear independence, T_1 can use zero-forcing matrices $\boldsymbol{N}_{12}$ and $\boldsymbol{N}_{13}$ of $d_{12} \times M_1$ and $d_{13} \times M_1$ dimensions, to zero-force the interference and to separate the two dedicated information signals. These zero-forcing matrices must satisfy:

$$\boldsymbol{N}_{12}\boldsymbol{H}_{13}(\boldsymbol{V}_{13} + \boldsymbol{V}_{23}) = \boldsymbol{0}_{d_{12} \times (d_{13} + d_{23})}, \tag{35}$$

$$\boldsymbol{N}_{13}\boldsymbol{H}_{12}(\tilde{\boldsymbol{V}}_{12} + \bar{\boldsymbol{V}}_{12} + \boldsymbol{V}_{32}) = \boldsymbol{0}_{d_{13} \times (d_{12} + d_{32})}. \tag{36}$$

Note that by zero-forcing $\boldsymbol{H}_{13}\boldsymbol{V}_{23}$, also $\boldsymbol{H}_{12}\boldsymbol{V}_{32}$ is zero-forced (and vice-versa) by (18). By using the proposed null-space beam-forming and zero-forcing, receiver T_1 obtains:

$$\boldsymbol{N}_{12}\boldsymbol{y}_1 = \boldsymbol{N}_{12}\boldsymbol{H}_{12}(\tilde{\boldsymbol{V}}_{12}\tilde{\boldsymbol{u}}_{12} + \bar{\boldsymbol{V}}_{12}\bar{\boldsymbol{u}}_{12}) + \boldsymbol{N}_{12}\boldsymbol{z}_1, \tag{37}$$

$$\boldsymbol{N}_{13}\boldsymbol{y}_1 = \boldsymbol{N}_{13}\boldsymbol{H}_{13}\boldsymbol{V}_{13}\boldsymbol{u}_{13} + \boldsymbol{N}_{13}\boldsymbol{z}_1. \tag{38}$$

Thus, T_1 recovers d_{12} linearly independent noisy observations of $\tilde{\boldsymbol{u}}_{12}$ and $\bar{\boldsymbol{u}}_{12}$, and also d_{13} linearly independent noisy observations of $\boldsymbol{u}_{13}$ as $\boldsymbol{N}_1 = [\boldsymbol{N}_{12}^{\mathsf{T}}\boldsymbol{N}_{13}^{\mathsf{T}}]^{\mathsf{T}}$ has sufficient row rank $d_{12} + d_{13}$, almost surely. Thus, T_1 can decode all dedicated signals and achieves a number of $d_{12} + d_{13}$ DoF.

On the receiver-side of T_2, we have:

$$\boldsymbol{y}_2 = \boldsymbol{H}_{21}[\tilde{\boldsymbol{V}}_{21}\ \bar{\boldsymbol{V}}_{21}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{21}\\ \bar{\boldsymbol{u}}_{21}\end{bmatrix} + [\boldsymbol{H}_{21}\tilde{\boldsymbol{V}}_{31}\tilde{\boldsymbol{u}}_{31} + \boldsymbol{H}_{23}\boldsymbol{V}_{13}\boldsymbol{u}_{13}] + \boldsymbol{H}_{23}\boldsymbol{V}_{23}\boldsymbol{u}_{23} + \boldsymbol{z}_2. \tag{39}$$

Note that $\bar{\boldsymbol{u}}_{31}$ is not observed by T_2 due to (20). Similarly to T_1, we need the following constraint to guarantee the linear independence of the desired signals and the interference:

$$0 \le \tilde{d}_{21} + \bar{d}_{21} + \tilde{d}_{31} + \tilde{d}_{23} \le M_2. \tag{40}$$

We use zero-forcing matrices $\boldsymbol{N}_{21}$ and $\boldsymbol{N}_{23}$ of $d_{21} \times M_2$ and $d_{23} \times M_2$ dimensions, respectively, satisfying:

$$\boldsymbol{N}_{21}\boldsymbol{H}_{23}(\boldsymbol{V}_{23} + \boldsymbol{V}_{13}) = \boldsymbol{0}_{d_{21} \times (d_{23} + d_{13})}, \tag{41}$$

$$\boldsymbol{N}_{23}\boldsymbol{H}_{21}(\tilde{\boldsymbol{V}}_{21} + \bar{\boldsymbol{V}}_{21} + \tilde{\boldsymbol{V}}_{31}) = \boldsymbol{0}_{d_{23} \times (d_{21} + \tilde{d}_{31})}, \tag{42}$$

to zero-force the interference and to separate the two dedicated information signals. By zero-forcing $\boldsymbol{H}_{23}\boldsymbol{V}_{13}$, also $\boldsymbol{H}_{21}\tilde{\boldsymbol{V}}_{31}$ is zero-forced (and vice-versa) by (22). With this scheme, receiver T_2 obtains:

$$\boldsymbol{N}_{21}\boldsymbol{y}_2 = \boldsymbol{N}_{21}\boldsymbol{H}_{21}(\tilde{\boldsymbol{V}}_{21}\tilde{\boldsymbol{u}}_{21} + \bar{\boldsymbol{V}}_{21}\bar{\boldsymbol{u}}_{21}) + \boldsymbol{N}_{21}\boldsymbol{z}_2, \tag{43}$$

$$\boldsymbol{N}_{23}\boldsymbol{y}_2 = \boldsymbol{N}_{23}\boldsymbol{H}_{23}\boldsymbol{V}_{23}\boldsymbol{u}_{23} + \boldsymbol{N}_{23}\boldsymbol{z}_2. \tag{44}$$

T_2 recovers d_{21} linearly independent noisy observations of $\tilde{\boldsymbol{u}}_{21}$ and $\bar{\boldsymbol{u}}_{21}$, and d_{23} linearly independent noisy observations of $\boldsymbol{u}_{23}$ from $\boldsymbol{y}_2$ since $\boldsymbol{N}_2 = [\boldsymbol{N}_{21}^{\mathsf{T}}\boldsymbol{N}_{23}^{\mathsf{T}}]^{\mathsf{T}}$ has sufficient row rank $d_{21} + d_{23}$, almost surely. Hence, T_2 achieves a number of $d_{21} + d_{23}$ DoF.

On the receiver-side of T_3, we have:

$$\boldsymbol{y}_3 = \boldsymbol{H}_{31}[\tilde{\boldsymbol{V}}_{31}\ \bar{\boldsymbol{V}}_{31}]\begin{bmatrix}\tilde{\boldsymbol{u}}_{31}\\ \bar{\boldsymbol{u}}_{31}\end{bmatrix} + [\boldsymbol{H}_{31}\tilde{\boldsymbol{V}}_{21}\tilde{\boldsymbol{u}}_{21} + \boldsymbol{H}_{32}\tilde{\boldsymbol{V}}_{12}\tilde{\boldsymbol{u}}_{12}] + \boldsymbol{H}_{32}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{z}_3. \tag{45}$$

At T_3, the signals $\bar{\boldsymbol{u}}_{21}$ and $\bar{\boldsymbol{u}}_{12}$ are not observed due to (24) and (26). We need the following constraint to guarantee the linear independence of the desired signals and the interference:

$$0 \le \tilde{d}_{31} + \bar{d}_{31} + \tilde{d}_{21} + \tilde{d}_{32} \le M_3. \tag{46}$$

We use zero-forcing matrices $\boldsymbol{N}_{31}$ and $\boldsymbol{N}_{32}$ of dimensions $d_{31} \times M_3$ and $d_{32} \times M_3$, satisfying:

$$\boldsymbol{N}_{31}\boldsymbol{H}_{32}(\boldsymbol{V}_{32} + \tilde{\boldsymbol{V}}_{12}) = \mathbf{0}_{d_{31}\times(d_{32}+\tilde{d}_{12})}, \tag{47}$$

$$\boldsymbol{N}_{32}\boldsymbol{H}_{31}(\tilde{\boldsymbol{V}}_{31} + \bar{\boldsymbol{V}}_{31} + \tilde{\boldsymbol{V}}_{21}) = \mathbf{0}_{d_{23}\times(d_{31}+\tilde{d}_{21})}, \tag{48}$$

to zero-force the interference space and to separate the two dedicated information signals. Receiver T_3 obtains:

$$\boldsymbol{N}_{31}\boldsymbol{y}_3 = \boldsymbol{N}_{31}\boldsymbol{H}_{31}(\tilde{\boldsymbol{V}}_{31}\tilde{\boldsymbol{u}}_{31} + \bar{\boldsymbol{V}}_{31}\bar{\boldsymbol{u}}_{31}) + \boldsymbol{N}_{31}\boldsymbol{z}_3, \tag{49}$$

$$\boldsymbol{N}_{32}\boldsymbol{y}_3 = \boldsymbol{N}_{32}\boldsymbol{H}_{32}\boldsymbol{V}_{32}\boldsymbol{u}_{32} + \boldsymbol{N}_{32}\boldsymbol{z}_3. \tag{50}$$

Thus, T_3 can recover d_{31} linearly independent noisy observations of $\tilde{\boldsymbol{u}}_{31}$ and $\bar{\boldsymbol{u}}_{31}$, and d_{32} linearly independent noisy observations of $\boldsymbol{u}_{32}$ from $\boldsymbol{y}_3$ since $\boldsymbol{N}_3 = [\boldsymbol{N}_{31}^\mathsf{T}\boldsymbol{N}_{32}^\mathsf{T}]^\mathsf{T}$ has sufficient row rank $d_{31} + d_{32}$, almost surely. Hence, T_2 can decode its dedicated signals and achieves $d_{31} + d_{32}$ DoF.

Assembling all constraints on the achievable DoF, yields:

$$\begin{aligned}
\tilde{d}_{32} = \tilde{d}_{23} &\le (M_2 + M_3 - M_1)^+,\\
\bar{d}_{31} &\le \min\{M_3, M_1 - M_2\},\\
\bar{d}_{21} &\le \min\{M_2, M_1 - M_3\},\\
\bar{d}_{12} &\le M_2 - M_3,\\
\tilde{d}_{12} + \bar{d}_{12} + \tilde{d}_{32} + \tilde{d}_{13} &\le M_1,\\
\tilde{d}_{21} + \bar{d}_{21} + \tilde{d}_{31} + \tilde{d}_{23} &\le M_2,\\
\tilde{d}_{31} + \bar{d}_{31} + \tilde{d}_{21} + \tilde{d}_{32} &\le M_3.
\end{aligned}$$

Note that real-valued DoF can be approximated by using signal-extensions over multiple time-slots [4], [10]. By maximizing d_Σ subject to these non-negative constraints, we get the maximum achievable sum-DoF of this scheme. This maximization is a linear optimization problem which can be solved by using the simplex method. The maximization yields a sum-DoF of $2M_2$. To verify this, we set:

$$\tilde{d}_{32} = \tilde{d}_{23} = (M_2 + M_3 - M_1)^+, \tag{51}$$

$$\bar{d}_{31} = \min\{M_3, M_1 - M_2\}, \tag{52}$$

$$\bar{d}_{21} = \min\{M_2, M_1 - M_3\}, \tag{53}$$

$$\bar{d}_{12} = M_2 - M_3. \tag{54}$$

This allocation satisfies all the DoF constraints above, and leads to $d_\Sigma = 2\tilde{d}_{32} + \bar{d}_{31} + \bar{d}_{21} + \bar{d}_{12} = 2M_2$, achieving (16).

Appendix

The derivation of the dimensions for the intersection subspaces is slightly generalized w. r. t. [11, Lem. 1].

Lemma 2. *If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are complex $N\times M_1$ and $N\times M_2$ random matrices, respectively, whose entries are drawn randomly i. i. d., then there exists a $(\min\{M_1, N\}+\min\{M_2, N\}-N)^+$-dimensional intersection subspace between the two column spaces of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, almost surely.*

Proof: Let an $N\times 1$ vector $\boldsymbol{q}$ lie in $\mathrm{span}(\boldsymbol{A}_1)\cap\mathrm{span}(\boldsymbol{A}_2)$. Then, there exists $\boldsymbol{q}_i \in \mathbb{C}^{M_i\times 1}$, with $i = 1, 2$, such that:

$$\boldsymbol{q} = \boldsymbol{A}_1\boldsymbol{q}_1 = \boldsymbol{A}_2\boldsymbol{q}_2. \tag{55}$$

In matrix form this yields:

$$\begin{bmatrix} \boldsymbol{I}_N & -\boldsymbol{A}_1 & \mathbf{0} \\ \boldsymbol{I}_N & \mathbf{0} & -\boldsymbol{A}_2 \end{bmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{pmatrix} = \boldsymbol{M}\boldsymbol{x} = \mathbf{0}. \tag{56}$$

Note that $\mathrm{rank}(\boldsymbol{A}_i) = \min\{M_i, N\}$ holds almost surely. We compute the dimension of $\mathrm{span}(\boldsymbol{A}_1) \cap \mathrm{span}(\boldsymbol{A}_2)$ by computing the dimension of the nullity of $\boldsymbol{M}$. Since:

$$\mathrm{rank}(\boldsymbol{M}) = \min\{2N, \min\{M_1, N\} + \min\{M_2, N\} + N\}$$

holds for i.i.d. matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ almost surely, we can conclude with the rank-nullity theorem of linear algebra, that:

$$\begin{aligned}
&\dim(\mathrm{null}(\boldsymbol{M}))\\
&= \min\{M_1, N\} + \min\{M_2, N\} + N - \mathrm{rank}(\boldsymbol{M})\\
&= (\min\{M_1, N\} + \min\{M_2, N\} - N)^+
\end{aligned} \tag{57}$$

holds, almost surely. ■

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