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Exercise 5

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Problem 1. *(Distribution of eigenvalues)* Use Gerschgorin's Theorem to find the smallest regions in which the eigenvalues of the matrix $\boldsymbol{A}$ are concentrated. Is $\boldsymbol{A}$ positive definite? Determine the smallest interval $[\lambda_{\min}, \lambda_{\max}]$ in which the real part of the eigenvalues are distributed.

$$\boldsymbol{A} = \begin{pmatrix} 10 & 0.1 & 1 & 0.9 & 0 \\ 0.2 & 9 & 0.2 & 0.2 & 0.2 \\ 0.3 & -0.1 & 5+i & 0 & 0.1 \\ 0 & 0.6 & 0.1 & 6 & -0.3 \\ 0.3 & -0.3 & 0.1 & 0 & 1 \end{pmatrix}$$

Gerschgorin's Theorem: Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, with entries a_{ij}, be given. For $i, j \in \{1, \dots, n\}$ let $R_i = \sum_{\substack{j=1\\ j\neq i}}^n |a_{ij}|$ and $C_j = \sum_{\substack{i=1\\ i\neq j}}^n |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries. Then every eigenvalue of $\boldsymbol{A}$ lies within at least one of the discs centered at a_{ii} with radius $\min\{R_i, C_i\}$.

Note that if one of the discs is disjoint from the others then it contains exactly one eigenvalue. If the union of m discs is disjoint from the union of the other $n-m$ discs then the former union contains exactly m and the latter $n-m$ eigenvalues of $\boldsymbol{A}$.

Problem 2. *(PCA in 2-dimensional space)* Suppose that for n samples, the sample covariance matrix $\mathbf{S}_n$ is given by

$$\mathbf{S}_n = \begin{pmatrix} 14 & -14 \\ -14 & 110 \end{pmatrix}.$$

a) Calculate the spectral decomposition $\mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{T}}$ of $\mathbf{S}_n$ by determining the matrices $\mathbf{V}$ and $\boldsymbol{\Lambda}$.

b) Determine the best projection matrix $\mathbf{Q}$ to transform the two-dimensional samples to a one-dimensional data.

c) Determine the residuum $\frac{1}{n-1} \max_{\mathbf{Q}} \sum_{i=1}^n \|\mathbf{Q}\mathbf{x}_i - \mathbf{Q}\bar{\mathbf{x}}_n\|^2$ for the above choice of $\mathbf{Q}$.