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Exercise 13

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Problem 1. *(Projection Matrix)* Let $\mathbf{X}$ be a matrix in $\mathbb{R}^{m\times n}$ such that $(\mathbf{X}^T\mathbf{X})$ is invertible. Show that $\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ is the projection matrix onto the image of $\mathbf{X}$.

Problem 2. *(Moore-Penrose pseudoinverse)*

Let $\mathbf{A}$ be a matrix in $\mathbb{R}^{m\times n}$. The matrix $\mathbf{B}$ in $\mathbb{R}^{n\times m}$ is called Moore-Penrose pseudoinverse of $\mathbf{A}$ if the following conditions are satisfied:

- $\mathbf{ABA} = \mathbf{A}$
- $\mathbf{BAB} = \mathbf{B}$
- $\mathbf{AB} = (\mathbf{AB})^T$
- $\mathbf{BA} = (\mathbf{BA})^T$

The existence of this matrix has been proved by Penrose, 1955.

a) Prove that Moore-Penrose pseudoinverse of $\mathbf{A}$, denoted by $\mathbf{A}^\dagger$ is unique.

b) If $\text{rk}(\mathbf{A}) = m$, then $\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$.

c) If $\text{rk}(\mathbf{A}) = n$, then $\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.

d) Consider the singular value decomposition of $\mathbf{A}$ given as $\mathbf{UDV}^T$ with $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$ and $\mathbf{D} \in \mathbb{R}^{m\times n}$, a diagonal matrix of the singular values of $\mathbf{A}$:

$$\mathbf{D} = \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

with $\mathbf{S} = \text{diag}(\sigma_1, \dots, \sigma_r)$ and $\sigma_i > 0$, $i = 1, \dots, r$. Show that $\mathbf{B} = \mathbf{V}\mathbf{D}^{+}\mathbf{U}^T$ is Moore-Penrose pseudoinverse of $\mathbf{A}$ where:

$$\mathbf{D}^\dagger = \begin{bmatrix} \mathbf{S}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T.$$

Problem 3. *(One-dimensional Linear Regression)* Given n samples of (x_i, y_i), consider the linear regression problem:

$$y_i = \vartheta_0 + \vartheta_1 x_i + \epsilon_i, \quad i = 1, \dots, n.$$

Find ϑ_0 and ϑ_1.