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Exercise 2
- Proposed Solution -

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Solution of Problem 1

a) Since $\mathbf{W} \succeq \mathbf{V}$, $\mathbf{W} - \mathbf{V}$ is non-negative definite. Therefore $\mathbf{x}^T(\mathbf{W} - \mathbf{V})\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, which means:

$$\mathbf{x}^T\mathbf{W}\mathbf{x} \geq \mathbf{x}^T\mathbf{V}\mathbf{x}.$$

Using Courant-Fischer theorem, it is known that:

$$\max_{S:\dim(S)=k} \min_{\mathbf{x}\in S;\|x\|_2=1} \mathbf{x}^T\mathbf{W}\mathbf{x} = \lambda_k(\mathbf{W}).$$

and

$$\max_{S:\dim(S)=k} \min_{x\in S;\|x\|_2=1} \mathbf{x}^T\mathbf{V}\mathbf{x} = \lambda_k(\mathbf{V}).$$

However $\mathbf{x}^T\mathbf{W}\mathbf{x} \geq \mathbf{x}^T\mathbf{V}\mathbf{x}$ implies that $\lambda_k(\mathbf{W}) \geq \lambda_k(\mathbf{V})$.

b) Since $\mathbf{W} \succeq \mathbf{V}$, $\mathbf{W} - \mathbf{V}$ is non-negative definite. Therefore $\mathbf{x}^T(\mathbf{W} - \mathbf{V})\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{x} = \mathbf{e}_i$ where $\mathbf{e}_i$ is ith canonical basis with all zero elements except the ith element equal to one. Namely $\mathbf{e}_i(j) = 0$ for $j \neq i$ and $\mathbf{e}_i(i) = 1$. For example:

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Therefore $\mathbf{e}_i^T(\mathbf{W} - \mathbf{V})\mathbf{e}_i = w_{ii} - v_{ii}$ and since $\mathbf{W} - \mathbf{V} \succeq 0$, $w_{ii} - v_{ii} \geq 0$. $v_{ii} \leq w_{ii}$, for $i = 1, \ldots, n$

c) Similar to the previous problem, choose the vector $\mathbf{e}_{ij}$ such that $\mathbf{e}_{ij}(k) = 0$ for $j \neq i, j$ and $\mathbf{e}_{ij}(i) = 1$ and $\mathbf{e}_{ij}(j) = -1$. For example:

$$\mathbf{e}_{23} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}$$

Since $\mathbf{W} - \mathbf{V} \succeq 0$, $\mathbf{e}_{ij}^T(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} \geq 0$, but:

$$(\mathbf{W} - \mathbf{V})\mathbf{e}_{ij} = \begin{bmatrix} (w_{1i} - v_{1i}) - (w_{1j} - v_{1j}) \\ (w_{2i} - v_{2i}) - (w_{2j} - v_{2j}) \\ \vdots \\ (w_{ni} - v_{ni}) - (w_{nj} - v_{nj}) \end{bmatrix}$$

and

$$\mathbf{e}_{ij}^T(\mathbf{W}-\mathbf{V})\mathbf{e}_{ij} = [(w_{ii}-v_{ii})-(w_{ij}-v_{ij})]-[(w_{ji}-v_{ji})-(w_{jj}-v_{jj})]$$
$$[w_{ii}+w_{jj}-2w_{ij}]-[v_{ii}+v_{jj}-2v_{ij}]\,.$$

Since $\mathbf{e}_{ij}^T(\mathbf{W}-\mathbf{V})\mathbf{e}_{ij} \geq 0$, it holds that: $v_{ii}+v_{jj}-2v_{ij} \leq w_{ii}+w_{jj}-2w_{ij}$.

d) From the second part of the exercise, $v_{ii} \leq w_{ii}$, for $i = 1,\ldots,n$. Therefore :

$$\text{tr}(\mathbf{V}) = \sum_{i=1}^{n} v_{ii} \leq \sum_{i=1}^{n} w_{ii} = \text{tr}(\mathbf{W}).$$

e) Note that $\det(\mathbf{V}) = \prod_{i=1}^{n} \lambda_i(\mathbf{V})$ and $\det(\mathbf{W}) = \prod_{i=1}^{n} \lambda_i(\mathbf{W})$. Using the first part of this exercise $\lambda_i(\mathbf{V}) \leq \lambda_i(\mathbf{W})$, for $i = 1,\ldots,n$. Since all eigenvalues are non-negative, it holds that $\det(\mathbf{V}) \leq \det(\mathbf{W})$.

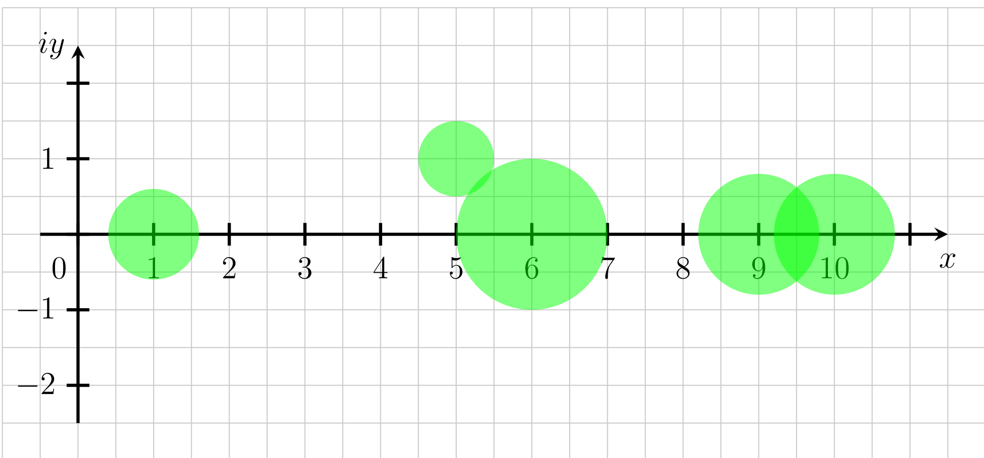
Solution of Problem 2

The radii $r_i = \min\{R_i, C_i\}$ of the discs are calculated by the aid of $R_i = \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|$ and $C_j = \sum_{\substack{i=1\\i\neq j}}^{n} |a_{ij}|$, and are given in the following table. The diagonal elements of $\boldsymbol{A}$ are the centers of the discs.

Table 1: The centers and radii of Gerschgorin's circles

i	a_{ii}	r_i	R_i	C_i
1	10	0.8	2.0	0.8
2	9	0.8	0.8	1.1
3	5+i	0.5	0.5	1.4
4	6	1.0	1.0	1.1
5	1	0.6	0.7	0.6

From the below figure we can observe that all areas of the circles are located on the right side of the plane. But having positive eigenvalues is not sufficient for $\boldsymbol{A}$ being positive definite. Since it is not symmetric, it will not be positive definite. Furthermore, we observe the limits $\lambda_{\min} = a_{55} - r_5 = 0.4$ and $\lambda_{\max} = a_{11} + r_1 = 10.8$. Note that since the disc located at a_{55} is disjoint from the others it contains exactly one of the eigenvalues.



Solution of Problem 3

(Weights on A Leverage)

A beam has niches with distances $d_1 \geq \cdots \geq d_n$ from the pivot. There are n weights of weight $w_1, \ldots, w_n$.

- The torque is calculated using the following equation:

$$\tau = \sum_{i=1}^{n} w_{f(i)} d_i,$$

where $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a bijective function. The weight $w_{f(i)}$ is placed in the niche i. Considered the ordered version of weights given by $w_{[1]} \geq w_{[2]} \geq w_{[n]}$. We have the following inequality:

$$\sum_{i=1}^{n} w_{f(i)} d_i \leq \sum_{i=1}^{n} w_{[i]} d_i.$$

We prove this using Abel's partial summation formula:

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n-1} (a_i - a_{i+1}) B_i + a_n B_n$$

where $B_i = \sum_{j=1}^{i} b_i$. For example see:

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = (a_1 - a_2) b_1 + (a_2 - a_3)(b_1 + b_2) + a_3 (b_1 + b_2 + b_3).$$

Applying this summation to $\sum_{i=1}^{n} w_{f(i)} d_i$, we have:

$$\sum_{i=1}^{n} w_{f(i)} d_i = \sum_{i=1}^{n-1} W_{f(i)} (d_i - d_{i+1}) + W_{f(n)} d_n,$$

with $W_{f(i)} = \sum_{j=1}^{i} w_{f(j)}$. On the other hand we have:

$$\sum_{i=1}^{n} w_{[i]} d_i = \sum_{i=1}^{n-1} W_{[i]} (d_i - d_{i+1}) + W_{[n]} d_n,$$

with $W_{[i]} = \sum_{j=1}^{i} w_{[j]}$.

Consider $W_{f(i)}$ and $W_{[i]}$. Since $W_{[i]}$ is the sum of i largest weights, we have:

$$W_{f(i)} \leq W_{[i]},$$

and since $d_i - d_{i+1} \geq 0$, we have:

$$(d_i - d_{i+1}) W_{f(i)} \leq (d_i - d_{i+1}) W_{[i]}.$$

This implies that:

$$\sum_{i=1}^{n-1} W_{f(i)} (d_i - d_{i+1}) + W_{f(n)} d_n \leq \sum_{i=1}^{n-1} W_{[i]} (d_i - d_{i+1}) + W_{[n]} d_n.$$

Hence,

$$\sum_{i=1}^{n} w_{f(i)} d_i \leq \sum_{i=1}^{n} w_{[i]} d_i.$$

Therefore the torque is maximized by putting the weights in an order on niches such that the largest one is on d_1 and decreasing afterward.

- For any given assignment of weights to niches, if the order follows the suggestion above, there is no room for improvement. Otherwise assume that for an assignment $w_{f(k)} < w_{f(j)}$ for $k < j$ and assume d_j's are different. Replacing these two weights will increase the torque. To see this, denote the new assignment by $f^*(\cdot)$ and see that:

  $$\sum_{i=1}^{n} w_{f^*(i)} d_i - \sum_{i=1}^{n} w_{f(i)} d_i = d_j(w_{f(k)} - w_{f(j)}) + d_k(w_{f(j)} - w_{f(k)}) = (w_{f(j)} - w_{f(k)})(d_k - d_j) > 0$$

  where the last inequality follows from the assumption $w_{f(k)} < w_{f(j)}$ and $d_k > d_j$ for $k < j$.