**Prof. Dr. Rudolf Mathar, Dr. Arash Behboodi, Markus Rothe**

# Exercise 5
# - Proposed Solution -

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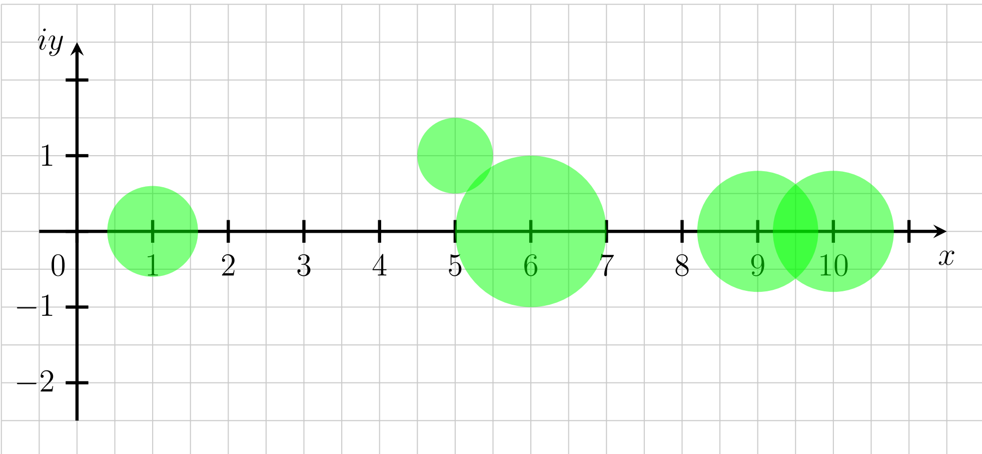
## Solution of Problem 1

The radii $r_i = \min\{R_i, C_i\}$ of the discs are calculated by the aid of $R_i = \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|$ and $C_j = \sum_{\substack{i=1\\i\neq j}}^{n} |a_{ij}|$, and are given in the following table. The diagonal elements of $\boldsymbol{A}$ are the centers of the discs.

Table 1: The centers and radii of Gerschgorin's circles

| $i$ | $a_{ii}$ | $r_i$ | $R_i$ | $C_i$ |
|---|---|---|---|---|
| 1 | 10 | 0.8 | 2.0 | 0.8 |
| 2 | 9 | 0.8 | 0.8 | 1.1 |
| 3 | 5+$i$ | 0.5 | 0.5 | 1.4 |
| 4 | 6 | 1.0 | 1.0 | 1.1 |
| 5 | 1 | 0.6 | 0.7 | 0.6 |

From the below figure we can observe that all areas of the circles are located on the right side of the plane. But having positive eigenvalues is not sufficient for $\boldsymbol{A}$ being positive definite. Since it is not symmetric, it will not be positive definite. Furthermore, we observe the limits $\lambda_{\min} = a_{55} - r_5 = 0.4$ and $\lambda_{\max} = a_{11} + r_1 = 10.8$. Note that since the disc located at $a_{55}$ is disjoint from the others it contains exactly one of the eigenvalues.




## Solution of Problem 2

**a)** The eigenvalues of $\mathbf{S}_n$ are solutions of $\det(\mathbf{S}_n - \mathbf{I}\lambda) = 0$. This leads to

$$\begin{vmatrix} 14-\lambda & -14 \\ -14 & 110-\lambda \end{vmatrix} = (14-\lambda)(110-\lambda)-14^2 = \lambda^2-124\lambda+1344 = (112-\lambda)(12-\lambda) = 0\,.$$

Hence, the diagonal matrix is determined by

$$\mathbf{\Lambda} = \begin{pmatrix} 112 & 0 \\ 0 & 12 \end{pmatrix}\,.$$

The eigenvectors $\mathbf{S}_n$ are solutions of $\mathbf{S}_n\mathbf{v} = \mathbf{v}\lambda$. In addition the eigenvectors should be normalized, i.e., $\|\mathbf{v}\| = 1$. We obtain

$$\begin{pmatrix} 14 & -14 \\ -14 & 110 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 112\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \Rightarrow \quad v_2 = -7v_1 \quad \text{and} \quad v_1^2 + v_2^2 = 1\,,$$

which yields the normalized eigenvector $\begin{pmatrix} \frac{1}{\sqrt{50}} & \frac{-7}{\sqrt{50}} \end{pmatrix}^{\mathrm{T}}$ for the eigenvalue 112. For the next eigenvector we only need to swap the entries of the first eigen vector and change the sign of one entry. This leads to the eigenvector $\begin{pmatrix} \frac{7}{\sqrt{50}} & \frac{1}{\sqrt{50}} \end{pmatrix}^{\mathrm{T}}$ for the eigenvalue 12. Putting the eigenvectors together we deduce the matrix

$$\mathbf{V} = \frac{1}{\sqrt{50}}\begin{pmatrix} 1 & 7 \\ -7 & 1 \end{pmatrix}\,.$$

**b)** The best projection matrix $\mathbf{Q}$ is determined by the first $k$ dominant eigenvectors $\mathbf{v}_i$ as $\mathbf{Q} = \sum_{i=1}^{k} \mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}$, where $k$ is the dimension of the image. For a transformation of a two-dimensional sample to a one-dimensional data (k=1), we obtain

$$\mathbf{Q} = \frac{1}{\sqrt{50}}\begin{pmatrix} 1 \\ -7 \end{pmatrix}\frac{1}{\sqrt{50}}\begin{pmatrix} 1 & -7 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 1 & -7 \\ -7 & 49 \end{pmatrix}\,.$$

**c)** The residuum $\frac{1}{n-1}\max\limits_{\mathbf{Q}}\sum_{i=1}^{n}\|\mathbf{Q}\mathbf{x}_i - \mathbf{Q}\bar{\mathbf{x}}_n\|^2$ is equal to the sum $\sum_{i=1}^{k}\lambda(\mathbf{S}_n)$ of dominant eigenvalues, that is equal to 112 in the present case.