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Tutorial 2

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Problem 1. *(Conditional Entropy)*

Let X, Y, Z be discrete random variables. Proof that:

a) $0 \overset{(i)}{\leq} H(X|Y) \overset{(ii)}{\leq} H(X)$.

- Equality holds in (i) $\Leftrightarrow$ X is totally dependent on Y.
- Equality holds in (ii) $\Leftrightarrow$ X and Y are independent.

b) $H(X|Y,Z) \leq \min\{H(X|Y), H(X|Z)\}$.

Problem 2. *(Sequence of Random Variables)*

Let $X_0 \in \mathcal{X}$ be a discrete random variable with distribution μ_0. Given the stochastic matrix Π, let $X_1, X_2, \ldots$ be the sequence of random variables (with the same support $\mathcal{X}$) such that μ_n is the distribution[1] of X_n and $\mu_n = \mu_{n-1}\Pi$ for all $n = 1, 2, \ldots$.

a) Show that $\mu_n = \mu_0 \Pi^n$.

Now assume that μ_n converges to some distribution μ^*, that is $\lim_{n\to\infty} \mu_n = \mu^*$ and $\mu^*\Pi = \mu^*$.

b) Proof that $D(\mu_n \| \mu^*) \geq D(\mu_{n+1} \| \mu^*)$ for all $n = 1, 2, \ldots$.

c) Show that if μ^* is the uniform distribution then $H(X_n) \leq H(X_{n+1})$ for all $n = 1, 2, \ldots$

Problem 3. *(A Metric)*

A function $\rho(x, y)$ is a metric if for all x, y,

- $\rho(x, y) \geq 0$,
- $\rho(x, y) = \rho(y, x)$,
- $\rho(x, y) = 0$ if and only if $x = y$,
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

[1] Note that μ_n are row vectors for all $n = 0, 1, 2, \ldots$.

a) Show that $\rho(X,Y) = H(X|Y) + H(Y|X)$ satisfies the first, second and fourth properties above. If we say that $X = Y$ if there is a one-to-one function mapping from X to Y, then the third property is also satisfied, and $\rho(X,Y)$ is a metric.

b) Verify that $\rho(X,Y)$ can also be expressed as

$$\begin{aligned}\rho(X,Y) &= H(X) + H(Y) - 2I(X;Y) \\ &= H(X,Y) - I(X;Y) \\ &= 2H(X,Y) - H(X) - H(Y)\end{aligned}$$

Problem 4. *(A Measure of Correlation)*

Let X_1 and X_2 be identically distributed random variables, but not necessarily independent. Let

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

a) Show that $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.

b) Show that $0 \leq \rho \leq 1$.

c) When is $\rho = 0$?

d) When is $\rho = 1$?

Problem 5. *(Entropy of a Sum)*

Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.

b) Give an example of (necessarily dependent) random variables in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

c) Under what conditions does $H(Z) = H(X) + H(Y)$?