Homework 5 in Cryptography I
- Proposal for Solution -

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Solution to Exercise 16.

Theorem 4.3 shall be proven.

(a) X is a discrete random variable with $p_i = P(X = x_i)$, $i = 1, \ldots, m$. It holds

$$H(X) = -\sum_i p_i \log p_i \geq 0,$$

as $p_i \geq 0$ and $-\log p_i \geq 0$ for $0 < p_i \leq 1$ and $0 \cdot \log 0 = 0$ per definition. Equality holds, if all addends are zero, i.e.,

$$p_i \log p_i = 0 \Leftrightarrow p_i \in \{0, 1\} \quad i = 1, \ldots, m,$$

as $p_i > 0$ and $-\log p_i > 0$, thus, $-p_i \log p_i > 0$ for $0 < p_i < 1$.

(b)

$$\begin{aligned} H(X) - \log m &= -\sum_i p_i \log p_i - \underbrace{\sum_i p_i}_{=1} \log m \\ &= \sum_{i:p_i>0} p_i \log \frac{1}{m\, p_i} \\ &= (\log e) \sum_{i:p_i>0} p_i \ln \frac{1}{m\, p_i} \\ &\overset{\ln z \leq z-1}{\leq} (\log e) \sum_{i:p_i>0} p_i \left(\frac{1}{m\, p_i} - 1 \right) \\ &= (\log e) \left(\sum_{i:p_i>0} \frac{1}{m} - 1 \right) \leq 0. \end{aligned}$$

As $\ln z = z - 1$ only holds for $z = 1$ it follows that equality holds iff $p_i = 1/m$, $i = 1, \ldots, m$. In particular, it follows $p_i > 0$, $i = 1, \ldots, m$.

(c) Define for $i = 1, \ldots, m$ and $j = 1, \ldots, d$

$$p_{i|j} = P(X = x_i \mid Y = y_j).$$

Show $H(X \mid Y) - H(X) \leq 0$ which is equivalent to the claim.

$$
\begin{aligned}
H(X \mid Y) - H(X) &= -\sum_{i,j} p_{i,j} \log p_{i|j} + \sum_i p_i \log p_i \\
&= -\sum_{i,j} p_{i,j} \log \frac{p_{i,j}}{p_j} + \sum_i \underbrace{\sum_j p_{i,j}}_{=p_i} \log p_i \\
&= \log(e) \sum_{i,j: p_{i,j}>0} p_{i,j} \ln \frac{p_i \, p_j}{p_{i,j}} \\
&\overset{\ln z \leq z-1}{\leq} \log(e) \sum_{i,j: p_{i,j}>0} p_{i,j} \left(\frac{p_i \, p_j}{p_{i,j}} - 1 \right) \\
&= \log(e) \left(\sum_{i,j: p_{i,j}>0} p_i \, p_j - 1 \right) \leq 0
\end{aligned}
$$

Note that from $p_{i,j} > 0$ it follows $p_i, p_j > 0$. Equality hold for $\frac{p_i \, p_j}{p_{i,j}} = 1$ which is equivalent to X and Y being stochastically independent.

This means that the transinformation $I(X, Y) = H(X) - H(X \mid Y)$ is nonnegative.

(d) It holds

$$
\begin{aligned}
H(X, Y) &= -\sum_{i,j} p_{i,j} \log p_{i,j} \\
&= -\sum_{i,j} p_{i,j} \left[\log p_{i,j} - \log p_i + \log p_i \right] \\
&= -\sum_{i,j} p_{i,j} \log \underbrace{\frac{p_{i,j}}{p_i}}_{p_{j|i}} - \sum_i \underbrace{\sum_j p_{i,j}}_{=p_i} \log p_i \\
&= H(Y \mid X) + H(X).
\end{aligned}
$$

(e) It holds

$$
H(X, Y) \overset{(d)}{=} H(X) + H(Y \mid X) \overset{(c)}{\leq} H(X) + H(Y)
$$

with equality as in (c) iff X and Y are stochastically independent.