Homework 3 in Optimization in Engineering

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Exercise 1. (Voronoi description) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to $\boldsymbol{a}$ than $\boldsymbol{b}$ is a halfspace, i.e.,

$$\left\{\boldsymbol{x} \,\middle|\, ||\boldsymbol{x}-\boldsymbol{a}||_2 \leq ||\boldsymbol{x}-\boldsymbol{b}||_2\right\} = \left\{\boldsymbol{x} \,\middle|\, \mathbf{c}^T\boldsymbol{x} \leq d\right\}$$

for a $\mathbf{c} \in \mathbb{R}^n_{\neq \mathbf{0}}$ and $d \in \mathbb{R}$. Now let $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$. The set

$$\mathcal{V} = \left\{\boldsymbol{x} \,\middle|\, ||\boldsymbol{x}-\boldsymbol{x}_0||_2 \leq ||\boldsymbol{x}-\boldsymbol{x}_i||_2,\ i = 1, \ldots, k\right\}$$

is called the *Voronoi region* around $\boldsymbol{x}_0$ with respect to $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$. From the above, it is a polyhedron as a finite intersection of halfspaces.

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron, given by a finite intersection of halfspaces. Show that there are points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k$ such that $\mathcal{P}$ is the Voronoi region around $\boldsymbol{x}_0$ with respect to $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

Exercise 2. (supporting hyperplanes) Represent each of the following closed, convex sets $\mathcal{C} \subseteq \mathbb{R}^2$ as an intersection of halfspaces.

i) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\|_2 \leq 1\}$, the 2-dimensional Euclidean unit ball with radius $r = 1$

ii) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^2 \,|\, x_2 \geq e^{x_1}\}$

iii) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^2 \,|\, x_2 \leq \log x_1,\ x_1 > 0\}$

iv) $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^2_{>0} \,|\, x_1 x_2 \geq 1\}$.