Homework 6 in Optimization in Engineering

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Exercise 1. (convex and concave functions) Decide which of the following functions are convex or concave and give reasons.

a) $f(x) = e^x - 1$, $x \in \mathbb{R}$

b) $f(\boldsymbol{x}) = x_1 x_2$, $\boldsymbol{x} \in \mathbb{R}^2_{>0}$

c) $f(\boldsymbol{x}) = \frac{1}{x_1 x_2}$, $\boldsymbol{x} \in \mathbb{R}^2_{>0}$

d) $f(\boldsymbol{x}) = e^{x_1^2 + x_2^2}$, $\boldsymbol{x} \in \mathbb{R}^2$

Exercise 2. (epigraph) Let $f \colon \mathcal{C} \to \mathbb{R}$ be a function defined on a convex, non-empty set $\mathcal{C} \subseteq \mathbb{R}^n$. Show that f is convex if and only if the epigraph of f

$$\operatorname{epi}(f) = \{(\boldsymbol{x}, y) \in \mathcal{C} \times \mathbb{R} \mid f(\boldsymbol{x}) \leq y\}$$

is a convex set.

Exercise 3. (separating convex and concave functions) Suppose $f \colon \mathbb{R}^n \to \mathbb{R}$ is a convex function and $g \colon \mathbb{R}^n \to \mathbb{R}$ is a concave function such that $g(\boldsymbol{x}) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x}$. Show that there exists an affine function $h \colon \mathbb{R}^n \to \mathbb{R}$ with $g(\boldsymbol{x}) \leq h(\boldsymbol{x}) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x}$. In other words, if a concave function g is an underestimator of a convex function f, then we can fit an affine function between f and g.