Homework 3 in Optimization in Engineering

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Exercise 1. (Separation Theorem) Complete the proof of the Separation Theorem (Theorem 2.1 in the lecture notes), which is proved for a special case in the lecture. Show that a separating hyperplane exists for two disjoint convex sets $\mathcal{C}$ and $\mathcal{D}$.

Hints.

- You can use the result proved in Theorem 2.1 in the lecture, i.e., that a separating hyperplane exists when there exist points in the two sets whose distance is equal to the distance between the two sets.
- If $\mathcal{C}$ and $\mathcal{D}$ are disjoint convex sets, then the set $\{\boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{x} \in \mathcal{C} , \boldsymbol{y} \in \mathcal{D}\}$ is convex and does not contain the origin.

Exercise 2. (Supporting hyperplanes) Represent each of the following closed, convex sets $\mathcal{C} \subseteq \mathbb{R}^2$ as an intersection of halfspaces.

(a) $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^2 \,|\, x_2 \geq e^{x_1}\}$.

(b) $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{R}^2_{>0} \,|\, x_1 x_2 \geq 1\}$.

Exercise 3. (Linear-fractional functions and convex sets) Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be the linear fractional function

$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^T\boldsymbol{x} + d}, \quad \mathbf{dom} f = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{c}^T\boldsymbol{x} + d > 0\}.$$

The inverse image of a convex set $\mathcal{C}$ under f is defined as

$$f^{-1}(\mathcal{C}) = \{\boldsymbol{x} \in \mathbf{dom} f \mid f(\boldsymbol{x}) \in \mathcal{C}\}.$$

Give a description of the inverse image $f^{-1}(\mathcal{C})$ for each of the following sets $\mathcal{C} \in \mathbb{R}^m$ as an intersection of the $\mathbf{dom} f$ with a halfspace in (a), with a polyhedron in (b), and with an ellipsoid in (c).

(a) The halfspace $\mathcal{C} = \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{g}^T\boldsymbol{y} \leq h\}$ with $\boldsymbol{g} \in \mathbb{R}^m_{\neq 0}$ and $h \in \mathbb{R}$.

(b) The polyhedron $\mathcal{C} = \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{G}^T\boldsymbol{y} \leq \boldsymbol{h}\}$ with $\boldsymbol{G} \in \mathbb{R}^m \times \mathbb{R}^n$ and $\boldsymbol{h} \in \mathbb{R}^n$.

(c) The ellipsoid $\mathcal{C} = \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{y}^T\boldsymbol{P}^{-1}\boldsymbol{y} \leq 1\}$ where $\boldsymbol{P} \in \mathcal{S}^m_{>0}$.