Homework 5 in Optimization in Engineering

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Exercise 1. (Products and quotients of convex functions) Let $f, g : \mathbb{R} \to \mathbb{R}_{>0}$. Prove the following statements.

(a) If f and g are convex and not decreasing (or not increasing), then the product $p(x) = f(x)g(x)$ is convex.

(b) If f and g are concave, f is not decreasing and g is not increasing (or vice versa), then the product $p(x) = f(x)g(x)$ is concave.

(c) If f is convex and not decreasing as well as g is concave and not increasing, then the quotient $q(x) = \frac{f(x)}{g(x)}$ is convex.

Exercise 2. (Geometric solution of optimization problem) Consider the optimization problem

$$\begin{aligned} &\text{minimize} \;\; f_k(\boldsymbol{x}) \\ &\text{subject to}\; g_i(\boldsymbol{x}) \leq 0, i = 1,2,3 \end{aligned}$$

for some $f_k, g_i : \mathbb{R}^2 \to \mathbb{R}$ with $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ and

$$\begin{aligned} g_1(\boldsymbol{x}) &= x_1^2 + x_2^2 - 1, \\ g_2(\boldsymbol{x}) &= x_1 + x_2 - 1, \\ g_3(\boldsymbol{x}) &= x_1 - x_2 - 1. \end{aligned}$$

(a) Plot the feasible set $M = \{\boldsymbol{x} \in \mathbb{R}^2 \mid g_i(\boldsymbol{x}) \leq 0,\; i = 1,2,3\}$.

(b) Solve the optimization problem geometrically for

$$\begin{aligned} f_1(\boldsymbol{x}) &= x_1^2 + x_2^2, \\ f_2(\boldsymbol{x}) &= (x_1 + 2)^2 + x_2^2, \\ f_3(\boldsymbol{x}) &= (x_1 - 2)^2 + x_2^2. \end{aligned}$$

Exercise 3. (Optimality criterion for convex problems) Let $f : \mathcal{C} \to \mathbb{R}$ with $\mathcal{C} \subseteq \mathbb{R}^n$ be a differentiable objective function of a convex optimization problem in standard form and $M[h, g] \subseteq \mathcal{C}$ the corresponding feasible set.

(a) Show that $\boldsymbol{x}^* \in M[h, g]$ is optimal, iff

$$\nabla f\left(\boldsymbol{x}^*\right)^T \left(\boldsymbol{y} - \boldsymbol{x}^*\right) \geq 0$$

for all $\boldsymbol{y} \in M[h, g]$.

(b) Let $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \boldsymbol{q}^T \boldsymbol{x} + r \ , \ \boldsymbol{x} \in \mathbb{R}^3$, with

$$\boldsymbol{P} = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix} , \qquad \boldsymbol{q} = \begin{pmatrix} -22 \\ -14.5 \\ 13.0 \end{pmatrix} , \qquad r = 1$$

be the objective function with constraints

$$-1 \leq x_i \leq 1 \ , \quad i = 1, 2, 3 \ .$$

Show that the point $\boldsymbol{x}^* = (1, 0.5, -1)^T$ is an optimal solution.

Hint: Use the first order condition for part (a).